VON NEUMANN REGULAR ELEMENTS LIFT MODULO ONE-SIDED NIL IDEALS; A DIRECT PROOF

Abstract. That idempotents, nilpotents and units lift modulo nil ideals is easy to prove (e.g., see 17. "Everything" lifts modulo nil ideals (2020), this Homepage, Elementary). The lifting of regular elements is just a little more difficult.

Definition. An element $x \in R$ is *regular modulo* I, where I is a one-sided ideal of R, if there exists some element $y \in R$ such that $x - xyx \in I$. We say that *regular elements lift* (or elements lift regularly) modulo I if whenever x is regular modulo I, there exists a regular element a in R such that $x - a \in I$ (also denoted $x \equiv a \mod I$).

Step 1. If $a^2 - a \in I$ where I is a nil right ideal (i.e., a is an idempotent modulo I), then $(a^2 - a)^m = 0$ for some $m \geq 1$. Multiplying by $(1 + a + \cdots + a^{m-1})^m$ gives $(a^{m+1} - a)^m = 0$.

Step 2. By the Binomial Theorem, $0 = (a - a^{m+1})^m = a^m(1 - a^m)^m = a^m(1 + ka^m) = a^m + k(a^m)^2$ for some $k \in R$. Since k commutes with a^m, it follows that $a^m \in a^m R a^m$, say $a^m x a^m = a^m$.

Step 3. *There is an idempotent $e \in aR$ such that $e - a \in I$.*

Indeed, from Step 2, $a^m(x + 1 - a^m x)a^m = a^m$, so define $e = a^m(x + 1 - a^m x) \in aR$. Hence $e^2 = e$ and $e - a^m \in (a^m - a^{2m})R$. But $a^2 \equiv a \pmod I$ so $a^k \equiv a^l \pmod I$ for all $k, l \geq 1$. It follows that $e \equiv a^m \equiv a \pmod I$, as required.

Step 4. Assume $x - xyx \in I$ for some $x, y \in R$. As xy is an idempotent modulo I, by Step 3 [xy takes the rôle of the above a], there exists an idempotent $e \in xyR$ such that $e - xy \in I$. Write $e = xyr$ for some $r \in R$. Note that $(ex)(yr)(ex) = e^3x = ex$ and so ex is regular. Finally, $ex \equiv xyx \equiv x \pmod I$, so x lifts regularly modulo I.

Remark. As the reader may have noticed, the above proof relies on two intermediate notions that are not made explicit (see [1]):

A one-sided ideal I of a ring R is said to be *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$, there is an idempotent $e \in xR$ such that $e - x \in I$, and

A one-sided ideal is called *π-regular* if some power of each element is regular.

Summarizing, the proof shows that π-regular one-sided ideals are strongly lifting, and, regular elements lift modulo strongly lifting one-sided ideals.

References

[1] W. K. Nicholson, Y. Zhou *Strong lifting*. J. Algebra **285** (2) (2005), 795-818.