# LZ-Reduced Abelian groups

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The term "reduced" was (and is) used is many senses in several domains of mathematics.

A type of modules designated as reduced modules are defined and investigated in [2] where a module $M$ is called *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $(mR) \cap (Ma) = 0$. Then

**Example (6)** [2] A module $M$ over $\mathbb{Z}$ is reduced iff, for any $m \in M$, either $m$ is torsion-free or the order of $m$ is square-free. In particular, for $n > 1$, $\mathbb{Z}_n$ is a reduced module over $\mathbb{Z}$ iff $n$ is square-free.

In what follows, the reduced $\mathbb{Z}$-modules (in the above sense) will be called *LZ-reduced* (short for Lee-Zhou).

By the example above, an Abelian group $G$ is LZ-reduced if its elements are of infinite order or square-free order.

Recall (e.g., [1]) that a (torsion Abelian) group is called *elementary* if it has only elements of square-free order. As such, it is a direct sum of cyclic groups of prime order.

Also recall that for an Abelian group $G$,

(i) the torsion part $T(G)$ is a pure subgroup of $G$,

(ii) (Kulikov, 1941) A bounded pure subgroup is a direct summand.

In particular, from (ii), it follows that if the torsion part of a group is elementary, then it is a direct summand of $G$.

Therefore

**Proposition 1** *For an Abelian group $G$, the following conditions are equivalent.*

*(i) $G$ is LZ-reduced,*

*(ii) the torsion part $T(G)$ is elementary,*

*(iii) $G$ is a direct sum of an elementary group any torsion-free group,*

*(iv) $G$ is a direct sum of cyclic groups of prime order and any torsion-free group.*

The customarily term for "reduced" Abelian group is for a group which has no divisible subgroups other than 0.

If $G$ is LZ-reduced then $G = E \oplus F$ with elementary $E$ and torsion-free $F$. Elementary groups are indeed reduced, but arbitrary torsion-free groups may not be reduced (these may have divisible subgroups, any direct sums of $\mathbb{Q}$).

## References

[1] L. Fuchs *Infinite Abelian Groups*. Volume 1, Academic Press, Pure and Applied Mathematics, vol. 36, 1970.

[2] T.K. Lee, Y. Zhou *Reduced modules*. in: Rings, modules, algebras, and abelian groups, Lecture Notes in Pure and Appl. Math. **236**, 365-377, Dekker, New York, 2004.