

Quasi-idempotent matrices

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An element $a \in R$ is said to *quasi-idempotent* if $q^2 = kq$ for some central unit k in R . We say $a \in R$ is *strongly regular* when $a \in a^2R \cap Ra^2$. Equivalently, $a \in R$ is strongly regular if and only if $a = ue = eu$, where $e^2 = e$ and u is a unit in R if and only if $a^2 = ua = au$ for some unit u in R .

Therefore, a quasi-idempotent is a special type of strongly regular element.

Lemma 1 *An element $q \in R$ is quasi-idempotent if and only if $q = ke$, where k is a central unit and e is an idempotent in R .*

Proof. If $q = ke$ then $q^2 = k(ke^2) = kq$ since e is idempotent. Conversely, assume $q^2 = kq$ for some central unit k . Multiplying by k^{-2} , shows that $e := k^{-1}q$ is idempotent. Hence $q = ke$. Both ways, the centrality of k is used. ■

Remarks. 1) As the identity is a central unit in any ring, all idempotent elements are quasi-idempotent.

2) If $2e \neq 0$ for an idempotent e in a ring R , then $-e$ is quasi-idempotent but not idempotent.

Proposition 2 *Let R be any unital ring. A matrix $A \in \mathbb{M}_n(R)$ is quasi-idempotent if there is a central unit $k \in R$ and an idempotent $E \in \mathbb{M}_n(R)$ such that $A = kE$.*

Proof. According to the previous lemma, it suffices to note that a central unit $K \in \mathbb{M}_n(R)$ is a scalar matrix kI_n with a central unit k of R . ■

References

- [1] Tang, G., Su, H. & Yuan, P. *Quasi-clean rings and strongly quasi-clean rings*. Communications in Contemporary Mathematics, 25 (02) (2023).