SOME GENERALIZATIONS OF REVERSIBLE RINGS

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ABSTRACT. Let X be a subset of a ring R. The ring R is said to be X-*reversible* if for any $a, b \in X$, the condition $ab = 0$ implies $ba = 0$. In this paper, we describe the X-reversible rings in the cases where X is the set of idempotents, the set of (unit)regular elements, or the set of quasiregular elements.

1. INTRODUCTION

Cohn [7] defined a ring R to be *reversible* if for any $a, b \in R$, the condition $ab = 0$ implies $ba = 0$. This concept has a rich history, as discussed in [1] or [14].

Anderson and Camillo [2], studied rings in which zero products commute, referring to such rings as ZC2, a term equivalent to what is now known as reversible. Prior to Cohn's work, these rings were investigated under different names: completely reflexive by Mason [17] and zero commutative by Habeb [10]. Later, Tuganbaev [19] explored reversible rings in his monograph on distributive lattices arising in ring theory, using the term of commutative at zero to describe these.

Clearly, commutative rings and reduced rings are reversible. A ring is called *Abelian* if every idempotent is central. It is easy to verify that every *reversible ring is Abelian*. Recently, various generalizations of reversible rings have been studied by many authors. The results obtained have found numerous applications in noncommutative ring theory.

To set the stage for our discussion, we begin with the following

Definition. Let X be a subset of a ring R. The ring R is called X-*reversible* if for any $a, b \in X$, the condition $ab = 0$ implies $ba = 0$. Obviously, that commutative rings are X-reversible for every subset X.

Clearly, the condition refers only to the (pairs of distinct) *zero divisors* which (both) belong to X. Since if $ab = 0$ then $(ba)^2 = 0$, both a and b are two-sided zero divisors. This also shows that (possibly noncommutative) reduced rings are X-reversible for every subset X.

In what follows, the term *pair* of zero divisors will be used exclusively in this sense.

Note that if a zero divisor a in R belongs to a subset X, i.e., $ab = 0$ for some $b \in R$, a zero divisor $c \in X$ with (say) $ac = 0$, may not exist.

To further generalize this definition, a ring R is said to be *trivially* X-reversible, if X contains no pairs of zero divisors. This way, *any domain D is trivially X-reversible*, for every subset X of D. As an example, the (noncommutative) domain of the Lipschitz quaternions $\mathbb{H}_{\mathbb{Z}}$ is trivially X-reversible for every subset X of $\mathbb{H}_{\mathbb{Z}}$.

In another direction, *every ring is trivially $U(R)$-reversible.*

For a ring R, $reg(R)$ denotes the set of all (von Neumann) regular elements of R, $ureg(R)$ denotes the set of all the unit-regular elements, $Id(R)$ denotes the set of all idempotents, $N(R)$ denotes the set of all nilpotent elements and $Q(R)$ denotes the set of all quasiregular elements. The Jacobson radical of R is denoted by $J(R)$.

For $X = N(R)$, the $N(R)$-reversible rings were studied in [1], under the term CNZ (commutation of nilpotents at zero).

In this paper, we describe the X-reversible rings for the cases $X = Id(R)$, $X = reg(R)$, $X = ureg(R)$ and $X = Q(R)$. It turns out that the $Id(R)$-reversible rings, the $reg(R)$-reversible rings and the $ureg(R)$-reversible rings are precisely the Abelian rings. The $Q(R)$-reversible rings, however, form some special subclass of nilpotent-reversible rings.

The rings we consider have identity and are nonzero (i.e., $1 \neq 0$). For a ring R, $Z(R)$ denotes the center of R.

2. General

Lemma 2.1. *Let $X, Y \subseteq R$.*

(i) If $X \subseteq Y$ and R is Y-reversible then R is also X-reversible.

(ii) If $0 \in X \cap Y$ and R is $(X+Y)$-reversible, then R is X-reversible and Y-reversible.

(iii) If $1 \in X \cap Y$ and R is XY-reversible, then R is X-reversible and Y-reversible.

(iv) If R is trivially X-reversible or trivially Y-reversible, then R may not be trivially XY-reversible, even if $XY = YX$.

(v) If R is reduced or commutative, then it is X-reversible for any subset X of R.

Proof. (i) Obvious.

(ii) As $0 \in X \cap Y$, it follows $X, Y \subseteq X + Y$ and we use (i).

(iii) As $1 \in X \cap Y$, it follows $X, Y \subseteq XY$ and we use (i).

(iv) for $X = Id(R)$ and $Y = U(R)$, it is well-known that $ureg(R) = XY = Id(R)U(R) = YX = U(R)Id(R)$, but $ureg(R)$ contains pairs of zero divisors (e.g., any nontrivial idempotent and the complementary idempotent).

(v) Just note that $ab = 0$ implies $(ba)^2 = 0$. □

Question. Can something special be proved if X is an ideal ? Example: $X = J(R)$.

3. New descriptions of Abelian rings

Recall that a ring R is called *idempotent reversible* if for every $e, f \in \mathrm{Id}(R)$, $ef = 0$ implies $fe = 0$.

An easy proof shows that this not a new class of rings.

Proposition 3.1. *A ring is idempotent reversible iff it is Abelian.*

Proof. The condition is obviously sufficient. Conversely, let $r \in R$ be arbitrary and $e^2 = e \in R$. The following (zero) product has idempotent factors: $\overline{e}(e + er\overline{e}) = 0$. By hypothesis, $(e + er\overline{e})\overline{e} = 0$ and so $er\overline{e} = 0$. Similarly, $(e + \overline{e}re)\overline{e} = 0$ gives $\overline{e}(e + \overline{e}re) = 0$ and so $\overline{e}re = 0$. Hence $er = ere = re$, as desired. □

We proceed with the following

Definitions. A (non-commutative) ring R is termed *regular reversible* if it is $reg(R)$-reversible and *unit-regular reversible* if it is $ureg(R)$-reversible. Since

$$Id(R) \subset ureg(R) \subset reg(R) \subset R$$

it follows that

$$reversible \Rightarrow reg.reversible \Rightarrow ureg.reversible \Rightarrow idemp.reversible = Abelian.$$

However, more can be proved.

Theorem 3.2. *Abelian rings are regular reversible.*

Proof. Let $a, b \in R$ for an Abelian ring R and let $a \in reg(R)$. There exists $x \in R$ such that $a = axa$. Suppose $ab = 0$. Then $xab = bxa = 0$ (as the idempotent xa is central). By right multiplication with a, we obtain $0 = bxa^2 = baxa = ba$ (again, xa is a central idempotent), as desired. □

Remark. As one can observe in the proof above, it suffices for only one of the elements $a, b \in R$ to be regular.

Corollary 3.3. *For any ring R, the following properties are equivalent.*
(i) R is regular reversible;
(ii) R is unit-regular reversible;
(iii) R is idempotent reversible.
(iv) R is Abelian.

4. Quasiregular reversible

Recall that an element r of a ring R is called *quasiregular* if $1 - r$ is a unit. We denote the corresponding set of elements by $Q(R)$.

It is well-known that $N(R) \subseteq Q(R)$. According to Lemma 2.1 (i), it follows that the $Q(R)$-reversible rings are nilpotent reversible.

Clearly, any statement about quasiregular elements admits an equivalent formulation in terms of the associated units.

A simple computation gives the following characterizations.

Theorem 4.1. *Let R be a ring. The following conditions are equivalent.*
(i) R is $Q(R)$-reversible;
(ii) for every $r, s \in Q(R)$, $rs = 0$ implies $sr = 0$;
(iii) for every units $u, v \in U(R)$, $(1 - u)(1 - v) = 0$ implies $(1 - v)(1 - u) = 0$;
(iv) for every units u, v of R, $1 + uv = u + v$ implies $uv = vu$.

Corollary 4.2. *The rings with commuting units (in particular, commutative rings) are quasiregular reversible.*

Remarks. 1) Note that the quasiregular reversibility requires commutation **only** for units with $1 + uv = u + v$.

This equality holds whenever $u = v$ is unipotent with a zerosquare nilpotent (i.e., $u = 1 + t$ with $t^2 = 0$).

This commutation clearly holds if $u = 1$ (and v is arbitrary) or $v = 1$ (and u is arbitrary). Moreover, it also holds for $u = v$.

2) The fact that *quasiregular reversible rings* are nilpotent reversible, follows also via unipotent elements, from the previous proposition.

4.1. **Matrix rings.**
Matrix rings that are quasiregular reversible are scarce.

Corollary 4.3. *Let R be a ring with* $\mathrm{char}(R) \neq 2$ *and let* $n \geq 2$ *be a positive integer. The matrix ring* $\mathbb{M}_n(R)$ *is not quasiregular reversible.*

Proof. We first prove the statement for $n = 2$. Take $U_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then $I_2 + U_2V_2 = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = U_2 + V_2$ but, if $2 \neq 0$, $U_2V_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \neq \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} = V_2U_2$. Hence, by Proposition 4.1, $\mathbb{M}_2(R)$ is not quasiregular reversible.

For an arbitrary $n > 2$, we take the $n \times n$ invertible matrices obtained by taking U_2 and V_2 respectively as 2×2 block left-upper corner and fill in with 1's on the diagonal and zeros elsewhere. For instance $U_n = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$. This way, $I_n + U_nV_n = U_n + V_n$ but $U_nV_n \neq V_nU_n$ and the proof is complete, using Proposition 4.1. □

Since the previous proof uses only upper triangular matrices, it also proves the corresponding result.

Corollary 4.4. *Let R be a ring with* $\mathrm{char}(R) \neq 2$ *and let $n \geq 2$ be a positive integer. The ring $\mathbb{T}_n(R)$ is not quasiregular reversible.*

Remarks. 1) In the remaining case, $\mathrm{char}(R) = 2$, we show that $\mathbb{M}_2(\mathbb{F}_2)$ is quasiregular reversible.

The ring has 6 units (incl. I_2), 3 of order two and 2 of order three. The quasiregular matrices are the 4 nilpotents (incl. 0_2), 3 of index two, namely, E_{12}, E_{21}, $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, and the 2 order three units, namely $\upsilon = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and $\omega = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Since $I_2 + \upsilon = \omega$ and $I_2 + \omega = \upsilon$, these units are quasiregular.

The only possible zero products xy of 2 quasiregular elements are either if $x = 0$ or $y = 0$, or else $x = y$ is zerosquare. In these cases, $yx = 0$ so the ring is quasiregular reversible.

However, $\mathbb{M}_2(\mathbb{F}_2)$ is not reversible (e.g., $E_{11}E_{21} = 0$ but $E_{21}E_{11} = E_{21} \neq 0$).

2) It follows from Theorem 4.3 that $\mathbb{M}_2(\mathbb{F}_3)$ is not quasiregular-reversible. A computer found 72 pairs of not quasiregular-reversible matrices. A sample follows.

Since $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ are units ($V^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$), $I_2 + U = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $I_2 + V = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ are quasiregular. Then $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 0_2$ but $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \neq 0_2$.

These results partly follow from [1].

There it is first proved that $\mathbb{T}_n(R)$ is not nilpotent-reversible for $n \geq 3$, using $E_{23}E_{12} = 0 \neq E_{13} = E_{12}E_{23}$. Then the closure to subrings shows that $\mathbb{M}_n(R)$ is also not nilpotent-reversible for $n \geq 3$. This shows that,

Theorem 4.5. *For any ring R and $n \geq 3$, both $\mathbb{T}_n(R)$ and $\mathbb{M}_n(R)$ are not quasiregular-reversible.*

Moreover, it is proved (see Theorem 2.7) that: *A ring R is reduced if and only if* $\mathbb{T}_2(R)$ *is nilpotent-reversible.*

Example 2.2 $\mathbb{M}_2(F)$ is nilpotent-reversible for any field F (not Abelian, and so not reversible). [By determining the nilpotent 2×2 matrices].

As already seen in the previous corollary, the equation $I_2 + UV = U + V$ for matrices, is of interest when studying quasiregular reversibility. For 2×2 matrices over commutative rings, a remarkable necessary condition can be found.

Proposition 4.6. *Let R be a commutative ring and $U, V \in \mathbb{M}_2(R)$. If $I_2 + UV = U + V$ then* $(\det(U) - 1)(\det(V) - 1) = (Tr(U) - 2)(Tr(V) - 2)$.

Proof. By taking determinants and traces, we find (only) some necessary conditions for the matrices U, V to satisfy the equation $I_2 + UV = U + V$.

Taking determinants:

$$\det(U + V) = \det(I_2 + UV) = 1 + Tr(UV) + \det(U)\det(V) \qquad (1).$$

Taking traces:

$$Tr(U) + Tr(V) = 2 + Tr(UV) \quad (2).$$

Next, we recall the special 2×2 formula

$$\det(U + V) + Tr(UV) = \det(U) + \det(V) + Tr(U)Tr(V) \qquad (3).$$

From (1) we get

$$\det(U + V) - Tr(UV) = 1 + \det(U)\det(V) \qquad (4).$$

Subtracting (4) from (3) we get

$$2Tr(UV) = \det(U) + \det(V) + Tr(U)Tr(V) - 1 - \det(U)\det(V),$$

and so

$$2(Tr(U)+Tr(V)-2) = \det(U)+\det(V)+Tr(U)Tr(V)-1-\det(U)\det(V) \qquad (5).$$

Equivalently,

$$(\det(U) - 1)(\det(V) - 1) = (Tr(U) - 2)(Tr(V) - 2) \qquad (6).$$

□

Delimit examples. We mentioned that for classes of rings, the following implications hold: commuting units ⇒ quasiregular reversible ⇒ nilpotent reversible.

A. *Quasiregular reversible ring but not commuting units.*

Since domains are trivially quasiregular reversible, any noncommutative domain that has two not commuting units will do. For example, the Lipschitz quaternions $\{a + bi + cj + dk \in \mathbb{H} : a, b, c, d \in \mathbb{Z}\}$, form a noncommutative domain (which is not a division ring) and i, j are two not commuting units.

As for a not trivially quasiregular reversible example, take $\mathbb{M}_2(\mathbb{F}_2)$ (see the previous remark). For instance, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are not commuting units.

B. *Nilpotent-reversible ring but not quasiregular reversible.*

The matrix ring $\mathbb{M}_2(k)$ for any field k (is clearly not Abelian but it) is nilpotent reversible (see [1]). According to Corollary 4.3, if char$(k) \neq 2$, $\mathbb{M}_2(k)$ is not quasiregular reversible.

4.2. **Factor rings.** In one direction, assume that R/I is quasiregular-reverible for some ideal I of R. If $1+uv = u+v$ holds for units u, v of R, then $\overline{1}+\overline{uv} = \overline{u}+\overline{v}$ implies $\overline{uv} = \overline{vu}$ and so only $uv - vu \in I$ follows.

Even if we *suppose that units lift modulo* I (e.g., I is nil), if $\overline{1}+\overline{uv} = \overline{u}+\overline{v}$ holds in R/I, by the lifting hypothesis, units u, v exist in R so that $1 + uv - u - v \in I$, not necessarily $= 0$. Hence $uv = vu$ may not follow.

Thus, *if R/I is quasiregular-reversible, then R may not be quasiregular-reversible.*

Example. Start with the ring $R = \mathbb{T}_n(S)$ for some ring S and some positive integer $n > 2$, which, according to Corollary 4.4, is not quasiregular-reversible. Take the ideal I which consists of the upper triangular matrices with zero diagonal entries. Then R/I is commutative so (quasiregular-) reversible.

In the opposite direction, if *R is quasiregular-reversible then R/I may not be quasiregular-reversible.*

Examples. 1) Let k be a field and let $R := k\langle x, y\rangle$ be the free associative k-algebra on two independent, noncommuting generators. Since R is a domain, R is quasiregular-reversible.

Let $I := (xy)$ be the two-sided ideal generated by xy, and set $\overline{R} := R/I$. In $\overline{R}$ we have $x \neq 0$, $y \neq 0$, but $xy = 0$ and $yx \neq 0$.

Define $u' := 1 + x$, $v' := 1 + y$ in $\overline{R}$. Both u' and v' are units, since their constant terms are equal to 1. Since $xy = 0$ in $\overline{R}$, a direct computation in $\overline{R}$ gives $1 + u'v' = u' + v'$ but $u'v' - v'u' = -yx \neq 0$, showing that $\overline{R}$ is not quasiregular-reversible.

2) In the same domain R, consider I the ideal generated by xy, x^2 and y^2 (see [1]). Then R/I is not nilpotent-reversible and so not quasiregular-reversible

4.3. **General.**

Proposition 4.7. *(1) The class of quasiregular-reversible rings is closed under direct products and (finite) direct sums.*

(2) Let $e \in R$ be a central idempotent. Then R is $Q(R)$-reversible iff eR and $(1-e)R$ are $Q(R)$-reversible rings.

Proof. (1) Follows at once from Theorem 4.1 since $U\left(\prod_{i\in I} R_i\right) = \prod_{i\in I} U(R_i)$ and similar for (finite) direct sums.

(2) It follows directly from (1) and the $Q(R)$-reversible closure for subrings, since $R \cong eR \oplus (1-e)R$. □

Proposition 4.8. *Let R be a ring. Then $R[x]$ is $Q(R)$-reversible iff so is R.*

Proof. Follows at once from Theorem 4.1 since $U(R[x]) = U(R)$. □

Proposition 4.9. *Let R be an algebra with identity over a commutative ring S. Then R is $Q(R)$-reversible iff the Dorroh extension $R^{\#}$ is $Q(R)$-reversible.*

Proof. Follows at once from Theorem 4.1 since $U(R^{\#}) \cong U(R) \times U(S)$ is a ring isomorphism given by $\phi(r, s) = (r + 1_R.s, s)$. □

5. The chart

Recall some well-known classes of rings which constitute an environment of our new classes.

Definitions. A ring is called *uni* ring (see [4]) if units commute with nilpotents, is *unit-central* if $U(R) \subseteq Z(R)$ and *nilpotent-central* if $N(R) \subseteq Z(R)$, the analogues of Abelian rings ($Id(R) \subseteq Z(R)$). The first class was studied in [13] and the second in [9] (as *CN*-rings) and (*central reduced* rings) in [20]. We just mention that there is a great deal of work on central units for integral group rings (see [12] for a comprehensive bibliography).

Finally, rings *with commuting units* and rings *with commuting nilpotents* (used in [5] and [13]) may be considered. The first were studied by various authors as rings with Abelian group of units.

Somehow related to the subject, we recall from [21] or [14], that a ring R is called *central reversible* if for any $a, b \in R$, $ab = 0$ implies ba belongs to the center of R (i.e., is a central zerosquare nilpotent). It is proved (Lemma 2.13) that *central reversible rings are Abelian.*

[Proof. Let $e^2 = e \in R$. For any $r \in R$, $\overline{e}(er - ere) = 0$ implies $(er - ere)\overline{e} = er - ere$ is central. Commuting $er - ere$ by e we have $er - ere = 0$.

Similarly for any $r \in R$, $(re - ere)\overline{e} = 0$ implies $re - ere = 0$. Therefore R is abelian.]

The converse fails: $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \equiv d (\mathrm{mod}\, 2), b \equiv c \equiv 0 (\mathrm{mod}\, 2) \right\}$.

Since for $ab = 0$, ba is zerosquare, it follows that *nilpotent central rings are central reversible.*

Recall that for every nilpotent element t in a ring with identity R, $1 + t$ is a unit which we call *unipotent*. Via unipotent elements, the following chart is readily checked (DF stands for Dedekind finite).

$$
\begin{array}{ccccccc}
 & & \text{commuting} - \text{units} & & & & \\
 & \nearrow & & \searrow & & & \\
\text{unit} - \text{central} & \rightarrow & \text{nilp} - \text{central} & \rightarrow & \text{uni} & \rightarrow & \text{commuting} - \text{nilp} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{central} - \text{reversible} & & \not\rightarrow & & \text{nilp} - \text{reversible} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{Abelian} & & \longrightarrow & & \text{DF}
\end{array}
$$

In the chart above, classes not connected by arrows are independent, and none of the arrows are reversible (examples gathered in [4]).

Based on the structure of this chart, we now turn our attention to the following two questions.

A. Which are the Abelian rings that are nilpotent reversible ? Are these the nilpotent-central rings (which - see chart - are Abelian and nilpotent reversible) ?

Negative: consider again $R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}) : a \equiv d (\mathrm{mod}\, 2), b \equiv c \equiv 0 (\mathrm{mod}\, 2) \right\}$.
Since R is a subring of $\mathbb{M}_2(\mathbb{Z})$, a matrix T of R is nilpotent iff $\det(T) = Tr(T) = 0$. From $ad - bc = 0$ follows that (not only b and c but also) a, d must be even, so $T = 2 \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ with $a^2 + bc = 0$. Hence T is the double of an arbitrary nilpotent matrix of $\mathbb{M}_2(\mathbb{Z})$. As $\mathbb{M}_2(\mathbb{Z})$ is *nilpotent reversible* (see [1] Lemma 2.4, (4)), so is

R (closure to subrings (2)). Moreover (see [20], Example 2.18), R is connected, so (trivially) *Abelian*, but *not nilpotent central*: $\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ is nilpotent but not central (not commuting with $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$).

The example shows that *even connected* rings (more special than the Abelian rings) that are nilpotent reversible, may not be nilpotent central. Thus, the characterization for Abelian and nilpotent reversible rings is not in the above chart and remains an open question.

We also mention (see [21], Example 2.14) that the ring R above, is *not* reversible nor *central reversible*: $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} = 0_2$ but $\begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \neq 0_2$ and $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ is not commuting with $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. So the possible arrow in the chart "central-reversible $\to$ nilpotent reversible", fails.

6. The quasiregular-reversible class

So far we have introduced two new classes of rings:
[commuting units $\Rightarrow$ quasiregular reversible $\Rightarrow$ nilpotent-reversible],
and in the chart we already had
[commuting units $\Rightarrow$ uni $\Rightarrow$ commuting nilpotents $\Rightarrow$ nilpotent-reversible].

The examples bellow show that the intermediate terms are pairwise independent.

1. Quasiregular reversible $\nRightarrow$ uni.

Indeed, according to Proposition 4.1, for every $u \in U(R)$, $t \in N(R)$, $1+(1+t)u = 1+t+u$ implies $(1+t)u = u(1+t)$, that is, $tu = ut$ whenever $tu = t$, so **not for every** u and t.

Example 1. Superseded by Example 3.

2. *Uni* $\nRightarrow$ *quasiregular reversible.*

Indeed, uni means that units commute **only** with unipotents, that is, even if $u + v = 1 + uv$, $uv = vu$ may fail, for units that are not unipotents.

[Recall that matrix rings are not uni]. Example 4 is not suitable, since that ring is also not uni.

Example. *Uni ring which is not units commuting* (example missing in [4]).

Every *reduced* ring is trivially uni. Hence, we can take the Lipschitz noncommutative domain $\mathbb{H}_{\mathbb{Z}}$. Indeed, $U(\mathbb{H}_{\mathbb{Z}}) = \{\pm 1, \pm i, \pm j, \pm k\}$ is the noncommutative quaternion group.

However, it is also quasiregular reversible: $(1-u)(1-v) = 0$ is possible only if $u = 1$ or $v = 1$. In both cases $uv = vu$ and we use Theorem 4.1.

Therefore, we take real quaternions $\mathbb{H}_{\mathbb{R}}$ which is a division ring, so reduced and trivially uni. Every $a \neq 1$ is quasiregular, as $1 - a$ is a unit. However, it is trivially X reversible, for any subset X.

Finally take the direct sum $R = \mathbb{H}_{\mathbb{R}} \times \mathbb{H}_{\mathbb{R}}$. Here $(i, 0)$ is quasiregular as $(1-i, 1)$ is a unit in R. Similarly, $(0, j)$. Then $(i, 0)(0, j) = 0$ but so is also $(0, j)(i, 0)$.

3. Quasiregular reversible $\nRightarrow$ *nilpotent commuting.*

Indeed, for two unipotents $1 + t$, $1 + x$, from Proposition 4.1, we get $tx = xt$ **only** if $tx = 0$.

Example 3. This supersedes Example 1, as uni $\Rightarrow$ nilpotent commuting.
We showed that $\mathbb{M}_2(\mathbb{F}_2)$ is quasiregular reversible. It is not nilpotent commuting: $E_{12}E_{21} = E_{11} \neq E_{22} = E_{21}E_{12}$.

4. Commuting nilpotents $\not\Rightarrow$ quasiregular reversible.
Indeed, commuting nilpotents refers only to units that are unipotents.
Example. Consider the ring $\mathbb{T}_2(R)$ of the upper triangular 2×2 matrices over a reduced ring of characteristic $\neq 2$. Then $N(\mathbb{T}_2(R)) = \begin{bmatrix} \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ is a zero-square ring and so trivially nilpotent commuting. According to Corollary 4.4, it is not quasiregular reversible.
This example is superseded by Example 2, as uni $\Rightarrow$ nilpotent commuting.

In Summary, we need only Examples 2 and 3 to show that the *newcomers are independent* from the old terms of the chart (i.e., those in [4]).

$$\begin{array}{ccccc} & & \text{quasireg} - \text{rev} & & \\ & \nearrow & & \searrow & \\ \text{comm} - \text{units} & & & & \text{nilpotent} - \text{rev} \\ & \searrow & & \nearrow & \\ & \text{uni} & \rightarrow & \text{comm} - \text{nilpotents} & \end{array}$$

In [4] it mentioned that for UU rings (i.e., rings with only unipotent units), commuting units, uni and commuting nilpotents are equivalent conditions.
It is easy to check that for UU rings, all six conditions above (including quasiregular reversible and nilpotent reversible) are equivalent conditions.

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