ON A CONJECTURE CONCERNING 3×3 INTEGRAL NIL-CLEAN MATRICES

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Abstract. We provide further evidence in support of the conjecture that every nil-clean 3×3 integral matrix is exchange.

1. Introduction

In [6], exchange elements (called *suitable* therein) were characterized in four equivalent ways. One of these descriptions states that an element a in a ring R is *left exchange* (or *left suitable*) if there exists an idempotent $e \in R$ such that $e - a \in R(a - a^2)$. It is known that every left exchange element is also right exchange, and conversely.

The relationships between nil-clean, clean, and exchange rings are well understood at the ring level. It is well known that nil-clean rings are clean [5], and that clean rings are exchange [6]. In particular, every clean element is exchange. However, the analogous implication for nil-clean elements is much subtler. In [5], Diesl posed the question whether every nil-clean element is clean. This was answered negatively in [1], where the matrix $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$ was shown to be nil-clean but not clean.

This naturally leads to the following question:

Are nil-clean elements necessarily exchange?

A counterexample to this question would in particular improve upon the example above by showing that nil-clean elements need not even satisfy the weaker exchange property.

Initial evidence suggested that the answer might be positive in small matrix rings. It was observed in [2] that all nil-clean integral 2×2 matrices, as well as the nil-clean elements in certain subrings of $\mathbb{M}_2(\mathbb{Z})$, are exchange. Consequently, any counterexample would need to be sought in larger matrix rings. Considerable effort in this direction was undertaken in [2, 3, 4], focusing mainly on $\mathbb{M}_3(\mathbb{Z})$ and several of its subrings, especially those with prescribed zero entries. These investigations led to the conjecture, strongly supported by computational evidence, that

Every nil-clean matrix in $\mathbb{M}_3(\mathbb{Z})$ is exchange.

In this short note, we provide another partial result toward the conjecture, namely a double negation argument. Starting with a non-exchange 3×3 matrix $A = [a_{ij}]_{1 \leq i,j \leq 3}$, we consider its *lower triangular part*, i.e., $\begin{bmatrix} a_{11} & . & . \\ a_{21} & a_{22} & . \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. If

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this partial matrix can be completed to an idempotent 3×3 matrix, then it yields a counterexample to the conjecture.

However, all non-exchange 3×3 matrices currently known fail to admit such an idempotent completion, and therefore none of them provides a counterexample to the conjecture.

2. The proof

Start with a non-exchange 3×3 matrix $A = [a_{ij}]$, $1 \leq i, j \leq 3$. Since all strictly upper triangular matrices are nilpotent, if there exists an idempotent completion $E = \begin{bmatrix} a_{11} & x_{12} & x_{13} \\ a_{21} & a_{22} & x_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then A is nil-clean and we have a counterexample for the conjecture, as $A = E + T$ with nilpotent $T = \begin{bmatrix} 0 & a_{12} - x_{12} & a_{13} - x_{13} \\ 0 & 0 & a_{23} - x_{23} \\ 0 & 0 & 0 \end{bmatrix}$.

To the best of our knowledge, only few non-exchange 3×3 integral matrices are currently known, all of them appearing in Corollary 5.3 [2].

Corollary 1. *The following* 3×3 *matrices are not exchange for any* $n \in \mathbb{Z}$, $n \geq 2$*:*

(a) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \begin{bmatrix} n\mathbb{Z}+1 & n\mathbb{Z} \\ n\mathbb{Z} & n\mathbb{Z}+1 \end{bmatrix}$, $\det(U) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0, 1\}$,

(b) $\begin{bmatrix} U & \mathbf{0} \\ \mathbf{0} & \varepsilon \end{bmatrix}$ *for* $U \in \mathbb{M}_2(n\mathbb{Z})$, $\det(U - I_2) \notin \{\pm 1\}$ *and* $\varepsilon \in \{0, 1\}$,

(c) $\begin{bmatrix} b & \mathbf{0} \\ \mathbf{0} & E \end{bmatrix}$ *with any* 2×2 *idempotent* E *and* $b \in \mathbb{Z} \setminus \{-1, 0, 1, 2\}$.

Our main result is

Theorem 2. *All* 3×3 *non-exchange matrices described in the previous corollary have a non-completable lower triangular part.*

Proof. Since we deal only with nonzero 3×3 idempotent matrices, the idempotent completions may have only traces 1 or 2. Accordingly, we consider completions with trace 2 or 1.

(a1) With trace $= 2$ ($\varepsilon = 0$):

$$\begin{bmatrix} 2n+1 & x & y \\ a & 1-2n & z \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2n+1 & x & y \\ a & 1-2n & z \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} . & 2x & . \\ 2a & ax + (1-2n)^2 & . \\ 0 & 0 & 0 \end{bmatrix} \text{ so}$$

$a = x = 0$ but $(1-2n)^2 \neq 1 - 2n$ (and $(2n+1)^2 \neq 2n+1$). Hence it is not completable.

(a2) With trace $= 1$ ($\varepsilon = 1$):

$$\begin{bmatrix} 2n+1 & x & y \\ a & -1-2n & z \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2n+1 & x & y \\ a & -1-2n & z \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} . & 0 & . \\ 0 & ax + (1+2n)^2 & . \\ 0 & 0 & 1 \end{bmatrix}$$

so $a = x = 0$ but $(1-2n)^2 \neq -1 - 2n$. Hence it is not completable.

(b) For $n \geq 2$, $U \in \mathbb{M}_2(n\mathbb{Z})$, $\begin{bmatrix} na & x & y \\ nb & nc & z \\ 0 & 0 & \varepsilon \end{bmatrix}$ is not completable:

$\begin{bmatrix} na & x & y \\ nb & nc & z \\ 0 & 0 & \varepsilon \end{bmatrix}^2 = \begin{bmatrix} . & n(a+c)x & . \\ . & . & . \\ 0 & 0 & \varepsilon \end{bmatrix}$ and $n(a+c)x = x$ implies $n(a+c) = 1$, which is impossible.

(c) If $b \neq 0, 1$ then $\begin{bmatrix} b & x & y \\ 0 & . & z \\ 0 & . & . \end{bmatrix}$ is not completable, as $b^2 \neq b$. □

Any hope of finding counterexamples to the conjecture using this new approach requires enlarging the current "library" of non-exchange 3 × 3 integral matrices.

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