ON MATRICES WITH PRESCRIBED ENTRIES

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ABSTRACT. Let n be a positive integer, and let A be a finite subset of a ring R. We say that A is idempotent (respectively, nilpotent, unit) n-completable if there exists an $n \times n$ idempotent (respectively nilpotent, invertible) matrix over R such that the elements of A appear among its entries.

In this work, we focus specifically on idempotent 3-completable triples, that is, sets of three elements that can be embedded into a 3×3 idempotent matrix. We prove that over any commutative unital ring, every triple is rank-one idempotent 3-completable. However, we also show that the analogous result fails for quadruples (e.g., over $\mathbb{Z}$).

1. INTRODUCTION

Idempotent matrices play a central role in linear algebra and ring theory, being closely related to projections, direct sum decompositions, and module-theoretic structures. Their classification, diagonalization, rank, and trace properties over various classes of rings have been extensively studied. By contrast, much less attention has been given to the following inverse-type problem:

given a finite subset of elements from a ring, when can these elements occur as entries of an idempotent matrix of fixed size?

The present paper is motivated by this inverse perspective. Given a positive integer n and a ring R, we say that a finite subset $A \subset R$ is *idempotent n-completable* if there exists an $n \times n$ idempotent matrix over R whose entries include all elements of A. Analogous notions can be defined for *nilpotent* and *invertible* matrices, as well as for *triangular* matrices. This terminology provides a natural and systematic framework for studying realizability questions for matrix entries over rings.

A first motivation of this work is that, despite the classical nature of idempotent matrices, the question of which finite sets of ring elements can be embedded into idempotent matrices appears to be largely unexplored. Even in small dimensions, the answer is far from obvious and depends in a subtle way on both algebraic and combinatorial constraints.

Our main focus is the case of idempotent 3-completability. The principal result of the paper establishes a surprising and strong positive statement: *over any commutative unital ring, every triple of elements is 3-completable by an idempotent matrix of rank (and trace) one.*

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This result is optimal. Indeed, we also show that the analogous statement fails for quadruples: over $\mathbb{Z}$, there exist four-element subsets which are not idempotent 3-completable. Thus, the paper identifies an exact boundary between universal completability and obstruction, highlighting a sharp transition from triples to quadruples.

A second motivation stems from the structural nature of the problem. While the notion of completability is algebraic, the placement of prescribed elements among the entries of a matrix is fundamentally a set-theoretic and combinatorial issue. For 3×3 matrices of rank one, there are precisely 84 distinct ways to assign three given elements to matrix entries. A complete understanding of the problem therefore requires a careful analysis of these configurations and the associated systems of equations. The proof combines algebraic techniques (divisibility conditions, Diophantine equations, rank considerations) with an exhaustive but structured case analysis.

A third motivation is methodological. Using known structural descriptions of rank-one idempotent matrices over GCD domains, we reduce the problem to the solvability of explicit systems of equations. Among the 84 possible configurations, we identify exactly two that are universally solvable for arbitrary triples, while all others admit explicit obstructions for suitable choices of ring elements. This approach not only yields the main theorem but also explains why the result holds and why it fails beyond triples.

Finally, the results of this paper naturally extend beyond the 3×3 case. As a consequence of the main theorem, we show that over any commutative unital ring, every n-tuple is n-completable by an idempotent matrix of rank one. Moreover, the negative results for quadruples suggest further questions concerning the maximal size of subsets that can be completed to idempotent, nilpotent, or invertible matrices in fixed dimension.

2. Simple results

Let m, n be positive integers and let the subset A of the ring R have m elements. Clearly, the definitions given in the Introduction, make sense only for $m \leq n^2$ for (full) matrices and $m \leq \dfrac{n(n+1)}{2}$ for upper triangular matrices. The following facts are straightforward:

a) the triangular completability implies the (full) completability;

b) if we have any completability for a pair (m, n), the completability for all pairs (k, n) where $k < m$, follows;

c) if $(a_1, ..., a_m)$ is completable, so is every permutation of it.

For some specific pairs (m, n) the n-completability is obvious.

Proposition 1. *(i) Every set of $\frac{n(n-1)}{2}$ elements in a ring is (triangular) unit n-completable and (triangular) nilpotent n-completable.*

(ii) If $m < n$, every m elements in a ring are (triangular) idempotent (or nilpotent or unit) n-completable.

Proof. (i) Indeed, $\frac{n(n-1)}{2}$ is the number of elements strictly above the diagonal of any $n \times n$ (triangular) matrix.

(ii) According to (b) above, it suffices to deal with the case $m = n - 1$. For the idempotent case, it suffices to take the nonzero elements $a_1, ..., a_{n-1}$ only in the first row, that is, $\begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \end{bmatrix}$, replace the 1 with 0, for the nilpotent case and add 1's on the diagonal, for the unit case. $\square$

Motivated by (ii), in what follows, we discuss the $m = n$ cases.

First the $m = n = 2$ case over commutative domains. This choice of base ring is motivated by the known structure of nontrivial idempotent (or nonzero nilpotent) 2×2 matrices, which have determinant zero and trace equal to 1 (resp. equal to 0).

Two elements a, b of a ring R are called *coprime* if there are elements s, t in R such that $as + bt = 1$.

Proposition 2. *(i) A pair of elements a, b of a commutative domain R is row (or column) idempotent 2-completable iff $b \mid a(1-a)$ (or $a \mid b(1-b)$), and is (anti)-diagonal completable iff $x(1-x) = ab$ is solvable in R.*

(ii) A pair of elements a, b of a commutative domain R is row (or column) nilpotent 2-completable iff $b \mid a^2$ (or $a \mid b^2$), and is diagonal completable iff $x^2 + ab = 0$ is solvable in R.

(iii) Every pair of elements a, b of a commutative ring R is diagonal unit 2-completable. It is row or column completable iff a, b are coprime.

Proof. (i) If the elements appear as entries of a row or a column, that is, in the form $\begin{bmatrix} a & * \\ b & 1-a \end{bmatrix}$ or its transpose, then 2-completability occurs iff $b \mid a(1-a)$. If the elements lie on the main diagonal, that is, $\begin{bmatrix} a & * \\ * & b \end{bmatrix}$, then 2-completability occurs iff $a + b = 1$ (equivalently $b = 1 - a$, which is a special of the previous condition). If the elements lie on the secondary diagonal, that is, $\begin{bmatrix} * & a \\ b & * \end{bmatrix}$, then 2-completability occurs iff the equation $x(1-x) = ab$ is solvable.

(ii) Analogous.

(iii) Indeed, $\det \begin{bmatrix} a & ab-1 \\ 1 & b \end{bmatrix} = 1$. For columns, for instance $\det \begin{bmatrix} a & x \\ b & y \end{bmatrix} = 1$ iff $ay - bx = 1$. Analogous for rows. $\square$

Remark. Over the integers, the last condition in (i) amounts to $1 - 4ab$ being an (odd) square and requires a, b to be of opposite signs.

Example. The 2-set $\{3, 8\}$ is not idempotent 2-completable over $\mathbb{Z}$. Indeed, by (i) above, $3 \nmid 8 \cdot 7$, $8 \nmid 3 \cdot 2$ and $x^2 - x + 24$ has no integer solutions. By (ii) above, it is also not nilpotent 2-completable.

Secondly, in the $m = n = 3$ case, (triangular) nilpotent and (triangular) unit 3-completability were mentioned in Proposition 1 (i).

Over commutative domains, in what follows, we describe the triangular idempotent 3-completability.

Lemma 3. *A triple a, b, c of nonzero elements of a commutative domain R is triangular idempotent 3-completable iff at least one of the following conditions hold.*

(a) At least one of a, b, c is equal to 1,

(b) $a, b, c \notin \{0, 1\}$ and $b = \pm ac$.

Proof. (a) Clearly (say $c=1$), $\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is a 3-completion.

(b) The (upper) triangular idempotent matrices are of form $\begin{bmatrix} * & a & b \\ 0 & * & c \\ 0 & 0 & * \end{bmatrix}$, where $*$ are idempotents, hence 0 or 1. With nonzero $a,b,c \neq 1$, a simple computation shows that only $\begin{bmatrix} 0 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{bmatrix}$ with $b=ac$, or $\begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$ with $b+ac=0$, are idempotent matrices. □

Example. The triple $(2,3,4)$ is *not* triangular idempotent 3-completable over many commutative domains, excepting for instance over $\mathbb{Z}_2$ or $\mathbb{Z}_3$.

3. The idempotent 3-completion of triples

The goal of this section is to prove the following surprising result. Recall that all over this section, the base ring is supposed to be commutative with identity.

Theorem 4. *Over any commutative unital ring, every triple is 3-completable by an idempotent matrix of rank (and trace) one.*

For the proof of this theorem, some prerequisites on idempotent matrices of rank (and trace) one are necessary.

A commutative ring R is called *GCD* if the greatest common divisor of every pair of elements of R exists. A GCD ring is called *Bézout* if the sum of two principal ideals is also a principal ideal. This means that Bézout's identity holds for every pair of elements.

First recall that the inner rank of an $m\times n$ matrix over a ring is the smallest integer r such that the matrix can be factored as a product of an $m\times r$ matrix and an $r\times n$ matrix.

Secondly, *over commutative Bézout domains*, the row rank, the column rank and the inner rank of any matrix are all equal (see Theorem 4.4, [3]). Therefore

Proposition 5. *Over any commutative Bézout domain, a 3×3 matrix of rank one is idempotent iff it has trace one.*

Proof. Indeed, since the rank and the inner rank coincide, Hence $E=\begin{bmatrix} a \\ b \\ c \end{bmatrix}\begin{bmatrix} s & t & v \end{bmatrix}$ and $Tr(E)=1$ iff $as+bt+cv=1$. It is easy to check that E is idempotent:

$$E^2=\begin{bmatrix} a \\ b \\ c \end{bmatrix}\underbrace{\begin{bmatrix} s & t & v \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}}\begin{bmatrix} s & t & v \end{bmatrix}=\begin{bmatrix} a \\ b \\ c \end{bmatrix}\cdot 1\cdot\begin{bmatrix} s & t & v \end{bmatrix}=E.$$

□

Denoting $C = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ we have the form $E = [sC, tC, vC]$ or, denoting $L = \begin{bmatrix} s & t & v \end{bmatrix}$, we have the form $E = \begin{bmatrix} aL \\ bL \\ cL \end{bmatrix}$.

Remark. The Bézout hypothesis on commutative domains can be further generalized to the so-called *Sylvester domains* (see [3], **5.5**).

To simplify the writing (and the forthcoming analyze), for the proof of this theorem we will search for completions using special forms for E: $E_3 = \begin{bmatrix} va_3 & va_2 & v(1-sa_2-va_3) \\ sa_3 & sa_2 & s(1-sa_2-va_3) \\ a_3 & a_2 & 1-sa_2-va_3 \end{bmatrix}$, or $E_2 == \begin{bmatrix} sa_3 & s(1-sa_3) & sa_1 \\ a_3 & 1-sa_3 & a_1 \\ 0 & 0 & 0 \end{bmatrix}$ or $E_3 = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, for some elements a_1, a_2, a_3, s, v of R. A similar structure also appears in [4].

Since E_3 involves only two variables, it is clearly insufficient to complete all triples in R^3.

For E_2, there are $\binom{6}{3} = 20$ possible ways to assign a, b, c to three of the six (nonzero) entries. To determine which pairs can be completed by some idempotent 3×3 matrix of the form E_2, we must solve 20 systems of equations involving the unknowns a_1, a_3 and s. Upon solving these systems, we found that even the E_2 form is insufficient to complete every triple in R^3. The result is presented below.

Proposition 6. *A triple is not (rank-one) idempotent 3-completable of the form E_2 iff it satisfies the following conditions:*

(i) None of the elements a, b, c divides another; that is, for any distinct pair $x, y \in \{a, b, c\}$, $x \nmid y$, $y \nmid x$.

(ii) For all $x, y \in \{a, b, c\}$, the value $1 - x$ does not divide y, and y does not divide $1 - x$.

(iii) For every ordered pair $(x, y) \in \{a, b, c\} \times \{a, b, c\}$ with $x \neq y$, the quadratic equation $xt^2 - t + y = 0$ with unknown t, is not solvable (that is, 6 such equations).

Proof. Here are just three samples from the 20 systems one has to solve.

$\begin{cases} sa_1 &= a \\ s(1-sa_3) &= b \\ sa_3 &= c \end{cases}$ is solvable iff $1 - c \mid b$ and $\dfrac{b}{1-c} \mid a$,

$\begin{cases} a_1 &= a \\ sa_1 &= b \\ s(1-sa_3) &= c \end{cases}$ is solvable iff $a \mid b$, $\dfrac{b}{a} \mid c$ and $\dfrac{b}{a} \mid 1 - \dfrac{ac}{b}$,

$\begin{cases} a_1 &= a \\ s(1-sa_3) &= b \\ a_3 &= c \end{cases}$ is solvable iff a is arbitrary and the equation $cs^2 - s + b = 0$ is solvable. $\square$

Examples. Over $\mathbb{Z}$, the following positive triples are not rank one idempotent 3-completable of the form E_2: $(4, 7, 10)$,$(5, 7, 9)$,$(6, 8, 11)$,$(7, 9, 11)$ but also (found by computer) $(22, 24, 47)$ or $(27, 34, 51)$.

What follows are *prerequisites* for **the proof of theorem 4**.

Let (a, b, c) be a triple that is idempotent 3-completed by a *rank one* idempotent E. Then the complementary (rank two) idempotent $I_3 - E$, 3-completes one of the following four triples: $(-a, -b, -c)$, $(1 - a, -b, -c)$, $(1 - a, 1 - b, -c)$ or $(1 - a, 1 - b, 1 - c)$.

The specific form depends on the placement of a, b, c in the diagonal of the rank one completion: if none of a, b, c lies on the diagonal, or only a lies on the diagonal, or a, b lie on the diagonal, or all of a, b, c are on the diagonal (then the trace $a + b + c = 1$), respectively. The converse also holds. If a triple (a, b, c) is idempotent 3-completed by a *rank two* 3×3 (nontrivial) idempotent F, then the complementary (rank one) idempotent $I_3 - F$, 3-completes one of the same four triples, determined analogously by the diagonal placement of a, b, c in the rank two completion.

Consequently, if there exists a triple that is *not* idempotent 3-completable by a *rank one* idempotent, then there must also exist a triple that is *not* idempotent 3-completed by a *rank two* idempotent. These non-completable triples correspond via the transformation above, depending on the diagonal configuration of the completion.

Rephrased, if a triple (a, b, c) is not rank one idempotent 3-completable, then at least one of the following triples is not rank two idempotent 3-completable: $(-a, -b, -c)$, $(1 - a, -b, -c)$, $(1 - a, 1 - b, -c)$ or $(1 - a, 1 - b, 1 - c)$. Thus, to determine all such non-completable triples, it suffices to focus on triples that are not rank one idempotent 3-completable.

Therefore, for the proof of Theorem 4, we concentrate on 3×3 idempotent matrices of the form E_1.

As noted in the Introduction, the location of three or four specified elements from a ring among the nine entries of a 3×3 matrix is not fundamentally an algebraic problem, but rather a set theoretic problem. This is why a careful analysis of all 84 possible configurations was unavoidable.

Specifically, for matrices of form E_1, there are $C_9^3 = 84$ distinct ways to assign the elements a, b, c to three of the nine entries. To determine which of these configurations can be completed by some idempotent 3×3 matrix of the form E_2, we must solve 84 different systems of equations, each involving the unknowns a_1, a_3 and s, v.

For the positive result - that every triple is rank one idempotent 3-completable - it suffices to find just one system (among the 84) that is solvable for all a, b, c in the base ring. In fact, we identified two such systems that meet this criterion.

For the negative result - that there exist quadruples that are not rank one idempotent 3-completable - it was essential to show that each of the $84-2 = 82$ remaining systems, is not solvable for some suitable choices of a, b, c in the base ring. Therefore it was necessary to examine each of these 82 systems individually and identify specific conditions on a, b, c that lead to unsolvability.

Each such system involves four variables denoted a_2, a_3, s, v consistent with the E_1 form of rank (and trace) one idempotents. Typically, the elimination of the variables a_2, a_3 yields a condition on the given a, b, c. If this condition is not satisfied, the corresponding system is not solvable, and the corresponding triple (or quadruple) is not idempotent 3-completable.

Since the complete analysis of all 84 systems spans eight pages, we provide below a few representative examples to illustrate the general method.

The 84 systems are associated to $E_1 = \begin{bmatrix} va_3 & va_2 & v(1-sa_2-va_3) \\ sa_3 & sa_2 & s(1-sa_2-va_3) \\ a_3 & a_2 & 1-sa_2-va_3 \end{bmatrix}$. In what follows we abbreviate the entries of E_1 like $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, in order to follow easily which cases are covered. For instance, 1) is 123, 2) is 124, ..., 7) is 129, 8) is 134, ..., 28) is 189, 29) will be 234, ... 49) will be 289, and so on: $28 + 21 + 15 + 10 + 6 + 3 + 1 = 84$.

There are six types of conditions on a, b, c which occur for the $82 = 84 - 2$ systems. We present 2 samples associated to each type.

A. 60 systems require divisibility conditions.

2-124) $\begin{cases} va_3 &= a \\ va_2 &= b \\ sa_3 &= c \end{cases}$. By multiplication, we get $a \mid bc$.

43-259) $\begin{cases} va_2 &= a \\ sa_2 &= b \\ 1-sa_2-va_3 &= c \end{cases}$. Here $va_3 = 1-b-c$ and $\frac{v}{s} = \frac{a}{b}$. Hence $sa_3a = b(1-b-c)$, i.e., $a \mid b(1-b-c)$.

B. 4 systems require solvability of some linear Diophantine equations.

1-123) $\begin{cases} va_3 &= a \\ va_2 &= b \\ v(1-sa_2-va_3) &= c \end{cases}$. By replacement, $v(1-a)-sb = c$, which is a linear Diophantine equation. It is solvable iff $\gcd(a-1, b) \mid c$.

65-456) $\begin{cases} sa_3 &= a \\ sa_2 &= b \\ s(1-sa_2-va_3) &= c \end{cases}$. Here $s(1-b)-va = c$, is a linear Diophantine equation that is solvable iff $\gcd(a, 1-b) \mid c$.

C. 7 systems require solvability of some quadratic Diophantine equations with zero constant term.

36-246) $\begin{cases} va_2 &= a \\ sa_3 &= b \\ s(1-sa_2-va_3) &= c \end{cases}$. By multiplying the third equation by v, and eliminating a_2, a_3, we get $as^2 - sv + bv^2 + cv = 0$, a quadratic Diophantine equation. The solution $(0,0)$ is not suitable.

74-489) $\begin{cases} sa_3 &= a \\ a_2 &= b \\ 1-sa_2-va_3 &= c \end{cases}$. We get a quadratic Diophantine $bs^2 - (1-c)s + av = 0$, whose solution $(0,0)$ is not suitable.

D. 6 systems require solvability of some quadratic Diophantine equations with nonzero constant term.

32-237) $\begin{cases} va_2 &= a \\ v(1-sa_2-va_3) &= b \\ a_3 &= c \end{cases}$. Here $v^2c - v + sa + b = 0$ is a quadratic Diophantine equation.

81-678) $\begin{cases} s(1-sa_2-va_3) & = & a \\ a_3 & = & b \\ a_2 & = & c \end{cases}$. A quadratic Diophantine equation: $cs^2+bsv-s+a=0$.

E. 3 systems require solvability of a quadratic equation with only one unknown.

24-168) $\begin{cases} va_3 & = & a \\ s(1-sa_2-va_3) & = & b \\ a_2 & = & c \end{cases}$. Here $s(1-sc-a)=b$ is a quadratic equation in s.

56-357) $\begin{cases} v(1-sa_2-va_3) & = & a \\ sa_2 & = & b \\ a_3 & = & c \end{cases}$. Here $v(1-b-vc)=a$ is a quadratic equation in v.

F. 2 systems require solvability of degree 3 equations with 2 unknowns. Using divisors of a or b, these reduce to quadratic Diophantine equations.

59-367) $\begin{cases} v(1-sa_2-va_3) & = & a \\ s(1-sa_2-va_3) & = & b \\ a_3 & = & c \end{cases}$. As $\frac{v}{s}=\frac{a}{b}$, replacing $s=\frac{bv}{a}$ in the first equation gives a degree 3 equation in v and a_2, namely $av(1-vc)-bv^2a_2=a^2$.

60-368) $\begin{cases} v(1-sa_2-va_3) & = & a \\ s(1-sa_2-va_3) & = & b \\ a_2 & = & c \end{cases}$. Analogous with 59: again $\frac{v}{s}=\frac{a}{b}$, replacing $v=\frac{sa}{b}$ in the second equation gives a degree 3 equation in s and a_3: $bs(1-sc)-as^2a_3=b^2$.

Remarks. 1) The degree 3 equations in 59-60 are of form $\alpha x^2y+\beta\gamma x^2+\beta x=\beta^2$ with $x\mid\beta$, or $\alpha x^2y+\beta x(1+\gamma x)=\beta^2$. We can cancel x and browse the divisors of β, for a quadratic Diophantine equation (with nonzero constant term).

2) The 2 systems which are solvable for every a,b,c are (17-148) and (41-257) and any of these can be used for the proof below.

3) If we have a quadratic Diophantine equation with a square discriminant (Δ), this often suggests that the equation is "factorable". However, even with the constant term $c=0$, it is still possible to choose the remaining coefficients so that the equation has no integer solutions. This is typically achieved by carefully selecting constants (such as f) so that the equation has no solutions modulo some integer - a technique known as *modular obstruction* (see [1]).

In many of the seven systems discussed above (specifically of type **C**), we have $c=d=f=0$, so the discriminant becomes $\Delta=b^2$ and the variables reduce to $x=s$, $y=v$. However, for an equation of the form $ax^2+bxy+ey=0$, when considered as quadratic in x, the discriminant is $D=(by)^2-4aey$. To ensure that the equation has only the trivial solution, the discriminant should not be a perfect square, which can be arranged through a suitable choice of a,b and e.

Proof. (continuation for **Theorem 4**). First observe that whenever one (or more) of the components a,b,c is (are) zero, say $c=0$, then by Proposition 1, the triple $(a,b,0)$ admits the idempotent completion $\begin{bmatrix} 1 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Henceforth we shall assume $a,b,c\neq 0$.

Secondly, if a, b, c are not different, it is easy to provide idempotent completions over any (not necessarily commutative) ring.

If $a = b = c$ then $E = \begin{bmatrix} a & a & a \\ 1 & 1 & 1 \\ -a & -a & -a \end{bmatrix}$ (or any permutation of rows) is an idempotent completion and

if $a = c$ then $E = \begin{bmatrix} a & a & a \\ b & b & b \\ -a-b+1 & -a-b+1 & -a-b+1 \end{bmatrix}$ (or any permutation of rows) is an idempotent completion.

It remains to discuss the case of triples with three different (nonzero) components.

One solution is given by (17-148), i.e., $\begin{cases} va_3 &= a \\ sa_3 &= b \\ a_2 &= c \end{cases}$. Here $a_3 = 1$, $v = a$, $s = b$ and $a_2 = c$ is a solution over any commutative ring. This means that every triple (a, b, c) is rank one idempotent 3-completable of form E_1 by the 3×3 matrix $E_{abc} = \begin{bmatrix} a & ac & a(1-a-bc) \\ b & bc & b(1-a-bc) \\ 1 & c & 1-a-bc \end{bmatrix}$. One can check $rank(E_{abc}) = trace(E_{abc}) = 1$ and $E_{abc}^2 = E_{abc}$.

Alternatively, another solution is given by (41-257), i.e., $\begin{cases} va_2 &= a \\ sa_2 &= b \\ a_3 &= c \end{cases}$. Here $a_2 = 1$, $v = a$, $s = b$ and $a_3 = c$ is a solution over any commutative ring. This means that every triple (a, b, c) is rank one idempotent 3-completable of form E_1 by the 3×3 matrix $E'_{abc} = \begin{bmatrix} ac & a & a(1-b-ac) \\ bc & b & b(1-b-ac) \\ c & 1 & 1-b-ac \end{bmatrix}$. One can check, $rank(E'_{abc}) = trace(E'_{abc}) = 1$ and $E'^2_{abc} = E'_{abc}$. □

Additionally, here are some special idempotent 3-completions over $\mathbb{Z}$.

Examples. 1) Suppose $a < b < c$ are integers (not necessarily positive) and say $b \equiv 1 \pmod{a}$. Then (a, b, c) is idempotent 3-completable over $\mathbb{Z}$, for any c.

Indeed, denote $b = aq + 1$. Then an idempotent 3-completion for the triple (a, b, c) is $E = \begin{bmatrix} b & a & c \\ -bq & -aq & -cq \\ 0 & 0 & 0 \end{bmatrix}$.

2) Every integral triple which includes 2 or -2 and an odd component, is idempotent 3-completable.

Indeed, the triple $(2, 2k+1, c)$ is completable to $\begin{bmatrix} 2k+1 & 2 & c \\ -k(2k+1) & -2k & -ck \\ 0 & 0 & 0 \end{bmatrix}$, for arbitrary c and for $(-2, 2k+1, c)$, a completion is $\begin{bmatrix} 2k+1 & -2 & c \\ k(2k+1) & -2k & ck \\ 0 & 0 & 0 \end{bmatrix}$.

3) The triples $(2a, 2b, 2c)$ are idempotent 3-completable if $2b - c = 1$, for arbitrary a.

A completion is $\begin{bmatrix} -c & -b & -a \\ 2c & 2b & 2a \\ 0 & 0 & 0 \end{bmatrix}$.

4) The triples $(\pm b, \pm b, \pm bn)$ (e.g., $a = -b$, $c = bn$) are idempotent 3-completable for any b and n (not necessarily even).

A completion is $\begin{bmatrix} -bn & -bn & -b \\ bn & bn & b \\ n & n & 1 \end{bmatrix}$.

5) The triples $(2, 2b, 4b)$ are idempotent 3-completable.

A completion is $\begin{bmatrix} 4b & 2b & 2 \\ -8b & -4b & -4 \\ 2b & b & 1 \end{bmatrix}$.

Due to the special form of the rank one idempotent 3-completion in the previous proof, it is now easy to cover also the general $n \times n$ case.

Theorem 7. *Over any commutative unital ring, every n-tuple is n-completable to an idempotent matrix of rank (and trace) one.*

Proof. Indeed, a completion of $(a_1, ..., a_n)$ is the $n \times n$ matrix
$$\begin{bmatrix} a_1 & a_1a_n & 0 & \cdots & 0 & a_1(1-a_1-a_2a_n) \\ a_2 & a_2a_n & 0 & \cdots & 0 & a_2(1-a_1-a_2a_n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-1}a_n & 0 & \cdots & 0 & a_{n-1}(1-a_1-a_2a_n) \\ 1 & a_n & 0 & \cdots & 0 & 1-a_1-a_2a_n \end{bmatrix}$$, with only three nonzero columns. □

Finally, we state and prove our second main result.

Theorem 8. *The maximum number of (distinct) elements in a subset of a ring that are idempotent* 3*-completable is* 3*. There are 4 different integers that are not idempotent 3-completable over $\mathbb{Z}$.*

Indeed, Theorem 4 shows that this number is ≥ 3, over any commutative unital ring. For the reversed inequality over $\mathbb{Z}$, we use the paragraphs after Proposition 6.

Since only 2 systems were solvable for any a, b, c (described in the proof of Theorem 4), for idempotent 3-completability of quadruples (a, b, c, d), we have to check only the systems obtained by adding another equation (i.e., entry) to the "good" systems 17) and 41). This requires checking another $6 + 6 = 12$ systems, for non-solvability.

First for (17-148).

$$(1248)\begin{cases} va_3 &= a \\ sa_3 &= b \\ a_2 &= c \\ va_2 &= d \end{cases}. \text{ Here } c \mid d.$$

$$(1348)\begin{cases} va_3 &= a \\ sa_3 &= b \\ a_2 &= c \\ v(1-sa_2-va_3) &= d \end{cases}. \text{ Here } a \mid bd.$$

$$(1458)\left\{\begin{array}{rcl} va_3 & = & a \\ sa_3 & = & b \\ a_2 & = & c \\ sa_2 & = & d \end{array}\right. . \text{ Here } c \mid d.$$

$$(1468)\left\{\begin{array}{ccc} va_3 & = & a \\ sa_3 & = & b \\ a_2 & = & c \\ s(1-sa_2-va_3) & = & d \end{array}\right. . \text{ Here } b \mid ad.$$

$$(1478)\left\{\begin{array}{rcl} va_3 & = & a \\ sa_3 & = & b \\ a_2 & = & c \\ a_3 & = & d \end{array}\right. . \text{ Here } d \mid a, b.$$

$$(1489)\left\{\begin{array}{ccc} va_3 & = & a \\ sa_3 & = & b \\ a_2 & = & c \\ 1-sa_2-va_3 & = & d \end{array}\right. . \text{ Here } 1-sc-a=d \text{ so } c \mid 1-a-d.$$

Secondly for (41-257)

$$(1257)\left\{\begin{array}{rcl} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ va_3 & = & d \end{array}\right. . \text{ Here } c \mid d.$$

$$(2357)\left\{\begin{array}{ccc} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ v(1-sa_2-va_3) & = & d \end{array}\right. . \text{ Here } v(1-b-vc)=d \text{ is a quadratic equation}$$

in v (not solvable, if $D=(b-1)^2-4cd$ is not a square).

$$(2457)\left\{\begin{array}{rcl} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ sa_3 & = & d \end{array}\right. . \text{ Here } c \mid d.$$

$$(2567)\left\{\begin{array}{ccc} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ s(1-sa_2-va_3) & = & d \end{array}\right. . \text{ Here } b \mid ad.$$

$$(2578)\left\{\begin{array}{rcl} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ a_2 & = & d \end{array}\right. . \text{ Here } d \mid a, b.$$

$$(2579)\left\{\begin{array}{ccc} va_2 & = & a \\ sa_2 & = & b \\ a_3 & = & c \\ 1-sa_2-va_3 & = & d \end{array}\right. . \text{ Here } 1-b-vc=d \text{ so } c \mid 1-b-d.$$

However, for the negative result stated above, this is not sufficient. Indeed, recall that for our positive result we used some special form for the 3×3 idempotent (rank and trace equal to one), and not the general form given after Proposition 5: for instance $E=\begin{bmatrix} aL \\ bL \\ cL \end{bmatrix}$ for a row $L=\begin{bmatrix} s & t & v \end{bmatrix}$. That is, we have to reconsider the 84 systems written for 5 variables a_1, a_2, a_3, s, v of R, involving 6 variables a, b, c, s, t, v. We skip this thoroughly new analysis.

A computer search gave 4567, 5678, 6789, as quadruples for which there are no a, b, c, s, t, v. Suggested by computer, we can state the following

Conjecture 9. *Over $\mathbb{Z}$, each quadruple $n, n+1, n+2, n+3$ ($n \geq 4$) is not idempotent 3-completable.*

In closing, we state the corresponding general open problems.

Let R be a commutative unital ring and n a given positive integer > 3. Find the maximum number of (distinct) elements in a subset of R which are (triangular) unit, or nilpotent, or idempotent n-completable.

Added in proof. A referee pointed out that some more references could be added to this subject:

J.A. Dias da Silva, Matrices with prescribed entries and characteristic polynomial, Proceedings of the American Mathematical Society, Volume 45, Number 1, 1974.

and

X. Zhan, Completion of a partial integral matrix to a unimodular matrix, Linear Algebra and its Applications, Volume 414, 2006, 373 377.

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