Left or right simple rings

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In Ring Theory, a ring R is called *simple* if $\{0\}$ and R are its only two-sided ideals.

It is well-known that a simple ring may have proper left (or right) ideals.

The best example is the (full) matrix ring $\mathbb{M}_n(D)$ over any division ring D.

If we fix i zero columns, for some $1 \leq i < n$, the corresponding set of matrices is a proper left ideal; symmetrically, if we fix some zero rows we obtain a proper right ideal.

Hence, for an undergraduate student, the following is a natural

Question. Why don't we use the term *left (or right) simple* for a ring R, whose left (resp. right) ideals are only $\{0\}$ and R ?

The answer is: *every left (or right) simple ring is a division ring* ! So indeed, there is no need to introduce these two terms.

Proof. Suppose (say) R is "left" simple and let $0 \neq a \in R$. Then Ra is a nonzero left ideal and by hypothesis, $Ra = R$. Hence there exists $b \in R$ such that $ba = 1$, that is, a is left-invertible.

Further, from $ba = 1$ it follows that $b \neq 0$, so repeating the above argument, there exists $c \in R$ such that $cb = 1$. By right multiplication with a we get $a = cba = c$ so $ab = 1$ and a is a unit. ■

A symmetric proof works for "right" simple rings.