Idempotent reversible rings

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December 28, 2017

In [1], rings whose *nilpotent elements commute at zero* [i.e. if $ab = 0$ for $a, b \in N(R)$ implies $ba = 0$] are studied.

[Clearly *nilpotent commuting* rings are NCZ]. It is proved that $\{\text{NCZ}\} \subset \{\text{DF}\}$ but independent from Abelian rings.

It seems natural to check what happens if we replace nilpotent elements with idempotents.

Definition. A ring R is called *idempotent reversible* if for every $e, f \in \text{Id}(R)$, $ef = 0$ implies $fe = 0$ [$\text{Id}(R)$ denotes the set of all the idempotents in the ring R].

Unfortunately, an easy proof shows that this not a new class of rings.

Proposition 1 *A ring is idempotent reversible iff it is Abelian.*

Proof. The condition is obviously sufficient. Conversely, let $r \in R$ be arbitrary and $e^2 = e \in R$. Then the following (zero) product has idempotent factors: $\overline{e}(e + er\overline{e}) = 0$. By hypothesis, $(e + er\overline{e})\overline{e} = 0$ and so $er\overline{e} = 0$. Similarly, $(e + \overline{e}re)\overline{e} = 0$ gives $\overline{e}(e + \overline{e}re) = 0$ and so $\overline{e}re = 0$. Hence $er = ere = re$, as desired. ■

References

[1] A. M. Abdul-Jabbar, C. A. K. Ahmed, T. K. Kwak, Y. Lee *On commutativity of nilpotent elements at zero*. Commun. Korean Math. Soc. **32** (4) (2017), 811-826.