Fusible Abelian groups

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Exercise. Determine the Abelian groups G if *every nonzero element of G is a sum of a torsion element and a torsion-free element*, that is a sum of a finite order element and an infinite order element.

Solution. We show that the condition characterizes precisely the torsion-free (Abelian) groups.

Denote by $T(G)$ the torsion part of G. If G is torsion-free, every nonzero element a has the decomposition $a = 0+a$. Conversely, by contradiction, assume $a \in T(G)$ has the property, i.e., $a = b + c$ with $b \in T(G)$ and $c \notin T(G)$. Since $c = a - b \in T(G)$, we have a contradiction. Hence G is torsion-free.

Prompted by the definition of fusible elements in rings (see Ghashghaei and McGovern, [2]), in [1], the authors define for any unital ring R, the so-called fusible R-modules as follows.

Definition. Let R be a ring with identity, M a right R-module and $0 \neq m \in M$. Then, m is called *fusible* if it can be expressed as the sum of a torsion element and a torsion-free element in M. The module M is said to be *fusible* if every non-zero element of M is fusible.

From the above exercise it follows that *the fusible Abelian groups* (as $\mathbb{Z}$-modules) *are precisely the torsion-free groups.*

References

[1] I. Baydar, B. Ungor, S. Halicioglu, A. Harmanci *Fusible modules*. Hacet. J. Math. Stat. **53** (3) (2024), 714-723.

[2] E. Ghashghaei, W. Wm. McGovern *Fusible rings*. Comm. Algebra **45** (3) (2017), 1151-1165.