A FORMULA FOR 2×2 MATRICES OVER COMMUTATIVE RINGS

It is well-known that the determinant preserves multiplication but not addition while the trace (the sum of the diagonal entries) preserves addition but not multiplication. For square matrices A, B of the same size, $\det(AB) = \det(A)\det(B)$ but typically $\det(A+B) \neq \det(A) + \det(B)$, $Tr(A+B) = Tr(A) + Tr(B)$ but usually $Tr(AB) \neq Tr(A)Tr(B)$.

In fact, it is often easier to find counterexamples to these equalities than to find cases where they actually hold.

Because of these non-equalities, computing $\det(A+B)$ or $Tr(AB)$ can be quite challenging.

However, there is a kind of "magical" exception !

For 2×2 matrices over commutative rings, the following identity holds and is easy to verify

$$\det(A+B) + Tr(AB) = \det(A) + \det(B) + Tr(A)Tr(B).$$

"Magical" because, although $\det(A+B) \neq \det(A) + \det(B)$ and $Tr(AB) \neq Tr(A)Tr(B)$ in general, adding the left-hand sides and the right-hand sides of these two non-equalities yields a valid equality.

This formula, however, generally fails for (square) matrices of size $n \geq 3$.

Example. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ Then $\det(A) = -2$, $\det(B) = -2$, $Tr(A) = 5$ and $Tr(B) = 13$.

Then $A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$, $\det(A+B) = -8 \neq \det(A) + \det(B)$ and $AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$, $Tr(AB) = 69 \neq Tr(A)Tr(B)$.

However $\det(A+B) + Tr(AB) = -61 = \det(A) + \det(B) + Tr(A)Tr(B)$.