IDEMPOTENT 2×2 MATRICES OVER COMMUTATIVE RINGS

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Abstract. The purpose of this note is twofold. First, we establish necessary conditions for 2×2 matrices over arbitrary commutative rings to be idempotent, expresed in terms of their trace and determinant. Second, we demonstrate that, within the same framework, the rank conditions $rk(E) = Tr(E)$ and $rk(E) + rk(I_2 - E) = 2$ are neither necessary nor sufficient for a 2×2 matrix to be idempotent. Finally, an example shows that even if all (five) conditions hold, the matrix may not be idempotent.

1. Introduction

Much is known about 2×2 idempotent matrices over commutative domains. Apart from the trivial idempotents 0_2, I_2, every nontrivial idempotent matrix has trace $= 1$ and zero determinant. Consequently, these are of form $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$.

As such, over PIDs, it is easy to check that the rank $rk(E) = Tr(E)$ and $rk(E) + rk(I_2 - E) = 2$.

Moreover, also over commutative domains, a 3×3 matrix E over a GCD domain R is nontrivial idempotent if and only if $\det(E) = 0$, $rk(E) = Tr(E) = 1 + \frac{1}{2}(TrH^2(E) - Tr(E^2))$ and $rk(E) + rk(I_{3-} - E) = 3$ (see [2]).

Over commutative rings, without any additional hypothesis, the situation is different.

The goal of this note is twofold: to find a maximal set of necessary conditions in terms of trace and determinant of the 2×2 nontrivial idempotent matrices over arbitrary commutative rings and to show, in the same context, that the above mentioned rank conditions are neither necessary nor sufficient for a 2×2 matrix to be idempotent.

Finally, an example shows that even if we gather all these conditions, these are not sufficient for a 2×2 matrix to be idempotent.

It might seem that our motivation is debatable: after all, checking if a matrix is idempotent is simple, while computing its trace and determinant to verify certain relations is more tedious. Our purpose here, therefore, is mainly theoretical.

2. The rank of 2×2 matrices over commutative rings

We first recall (from [1]) the notion of rank, in particular, for 2×2 matrices, and present some examples.

Let $A \in \mathbb{M}_2(R)$ over a nonzero commutative ring R. For each $n \in \{1, 2\}$, $I_n(A)$ denotes the ideal generated by all $n \times n$ minors of A. Then

$$(0) \subseteq I_2(A) \subseteq I_1(A) \subseteq R.$$

Here $I_2(A) = \det(A)R$ and $I_1(A) = aR + bR + cR + (t-a)R$, denoting the trace by $t := Tr(A)$ and $A = \begin{bmatrix} a & b \\ c & t-a \end{bmatrix}$.

Accordingly

$$(0) = Ann_R(R) \subseteq Ann_R(I_1(A) \subseteq Ann_R(I_2(A) \subseteq Ann_R((0)) = R.$$

Then we recall the

Definition. The rank of A, hereafter denoted $rk(A)$, is $\{\max(s) : Ann_R(I_s(A) = (0)\}$.

From [1] (see **4.11** (d) +(e) and Exercise 5) and some simple consequences we summarize

Lemma 2.1. *(i)* $rk(A) = 0$ *iff* $Ann_R(I_1(A)) \neq (0)$ *[that is,* 0 *is the maximum integer* t *above] iff there exists a nonzero* $r \in R$ *such that* $ra = rb = rc = r(t-a) = 0$.

(ii) $rk(A) = 1$ *iff* $Ann_R(I_2(A)) \neq (0)$ *[that is,* 1 *is the maximum integer* t *above] iff there exists a nonzero* $r \in R$ *such that* $r\det(A) = 0$.

(iii) $rk(A) = 2$ *iff* $Ann_R(I_2(A)) = (0)$ *[that is,* 2 *is the maximum integer* t *above] iff* $\det(A)$ *is cancellable.*

(iv) $rk(A) < 2$ *iff* $\det(A)$ *is a zero divisor (incl.* $\det(A) = 0$*) [actually, if* $\det(A) = 0$ *then* $Ann_R(I_2(A)) = R$*].*

(v) If $\det(A) \in U(R)$ *then* $rk(A) = 2$.

(vi) If A *has at least an unit entry then* $I_1(A) = R$ *and so* $Ann_R(I_1(A)) = (0)$. *Hence* $rk(A) > 0$ *[if (say)* $a \in U(R)$ *from* $ra = 0$ *we get* $r = 0$*].*

(vii) If A *has an unit entry and zero divisor determinant then* $rk(A) = 1$.

Remark. The converse in (v), fails (unless cancellable = unit, if the ring is finite).

An *example* for (vii) is $A = E_{ij}$ for any i, j. Hence the idempotents E_{11}, E_{22} and the nilpotents E_{12}, E_{21}, all have rank 1.

Examples. 1) $A = 2I_2$ over $\mathbb{Z}_4$ is a nonzero matrix of rank zero.

2) Over $\mathbb{Z}_6$, [1].

(a) $A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$. All entries are zero divisors. Here $I_2(A) = 4R$, $I_1(A) = 2R$ and $Ann(4R) = Ann(2R) = 3R \neq (0)$. Thus $rk(A) = 0$.

Alternatively, there exist $2 \neq 0$ with all products by the entries equal zero.

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. All entries are zero divisors. Since $\det(A) = 0$, **4.11 (e)** implies $rk(A) < 2$. Since $I_1(A) = 2R + 3R = R$, $Ann(I_1(A)) = (0)$. Therefore $rk(A) = 1$.

(c) $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$. Then $\det(A) = 5 \in U(R)$. Therefore $rk(A) = 2$ by **4.11 (d)**.

3) $A = \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z}_6)$, $t = 4$, $d = 3$, but $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \neq A$.

Regarding the rank, since $\det(A) = 3$ is a zero divisor, $rk(A) < 2$. Next $I_1(A) = R + 3R = R$ and $I_2(A) = 3R$. Then $Ann(I_1(A)) = (0)$ and so $rk(A) = 1 \neq Tr(A)$.

3. About idempotent 2×2 matrices

First observe that according to Cayley-Hamilton's theorem,

$$(Tr(E) - 1)E = \det(E)I_2 \qquad (*)$$

is equivalent to $E^2 = E$.

Hence if $E^2 = E$ and $t = Tr(E) = 1$ then $d := \det(E) = 0$ and so $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$, the already mentioned form of idempotent matrices over commutative domains.

Note from the start that $\det^2(E) = \det(E)$, so the *determinant must be an idempotent* of R i.e.,

$$d^2 = d.$$

Moreover, taking traces from (*) (or taking $Tr(E^2) = Tr(E)$ with a bit of computation), we get

$$t(t-1) = 2d.$$

From $(Tr(E) - 1)E = \det(E)I_2$, if $E = [e_{ij}]$, we get

(5) $(t-1)e_{11} = e_{11}e_{22} - e_{12}e_{21} = (t-1)e_{22}$ and

(6) $(t-1)e_{12} = 0 = (t-1)e_{21}$.

From (5) we obtain

(7) $(e_{11} - 1)e_{11} = -e_{12}e_{21} = (e_{22} - 1)e_{22}$

and from (6) we obtain also

(8) $(t-1)(e_{12} \pm e_{21}) = 0$.

Hence $e_{12}, e_{21}, e_{11} \pm e_{22} \in Ann(t-1)$, all are zero divisors, if $t \neq 1$.

Multiplying (*) by E, we get $(t-1)E = dE$, or equivalently, $(t-d-1)E = 0_2$. Hence

(9) $(t-d-1)e_{ij} = 0$, for all $i, j \in \{1, 2\}$.

Here all entries of E are in $Ann(t-d-1)$, so, if $t - d \neq 1$, all entries are zero divisors (incl. e_{11}, e_{22}).

By taking determinants, we also get

(10) $(Tr(E) - \det(E) - 1)\det(E) = 0$, or

$$(t-1)d = d^2 = d$$

which implies

$$(t-2)d = 0.$$

Equivalent conditions (but not in terms of trace, determinant and rank) to $E^2 = E$, are obviously

(1) $e_{11}^2 + e_{12}e_{21} = e_{11}$,

(2) $e_{12}t = e_{12}$,

(3) $e_{21}t = e_{21}$,

(4) $e_{12}e_{21} + e_{22}^2 = e_{22}$.

The conditions (1)-(4) are necessary and sufficient, the other conditions (5)-(10) are only necessary for a 2×2 matrix E to be idempotent..

Summarizing, denoting $d = \det(E)$, $t = Tr(E)$, for an idempotent 2×2 matrix E, and assuming $t \neq 1$ and $t - d \neq 1$, all entries are zero divisors and the following equalities are necessary:

$$t(t-1)=2d,(t-2)d=0,d^2=d.$$

4. The $t=d+1$ case

For 2×2 matrices over commutative rings we can prove the following equivalence.

Proposition 4.1. *Let R be a commutative ring and let $A\in\mathbb{M}_2(R)$. Then $Tr(A)=\det(A)+1$ iff $\det(A-I_2)=0$.*

Proof. Note that for 2×2 matrices

$$\det(A-I_2)=\det(A)-Tr(A)+1.$$

Therefore the statement is straightforward. □

There is an analogous result for 3×3 matrices.

Proposition 4.2. *Let R be a commutative ring and let $A\in\mathbb{M}_3(R)$. Then $\frac{1}{2}(Tr^2(A)-Tr(A^2))-Tr(A)=\det(A)-1$ iff $\det(A-I_3)=0$.*

Proof. Indeed, for 3×3 matrices

$$\det(A-I_3)=\det(A)-\frac{1}{2}(Tr^2(A)-Tr(A^2))+Tr(A)-1$$

holds. □

For the general $n\times n$ case, one has to use the coefficients in the Cayley-Hamilton's theorem:

$$A^n+c_{n-1}A^{n-1}+\cdots+c_1A+(-1)^nI_n=0_n.$$

The coefficients c_i are given by the elementary symmetric polynomials of the eigenvalues of A. Using Newton identities, the elementary symmetric polynomials can in turn be expressed in terms of power sum symmetric polynomials of the eigenvalues: $s_k=\sum_{i=1}^{n}\lambda_i^k=Tr(A^k)$. Thus, we can express c_i in terms of the trace of powers of A.

An explicit formula follows

$$c_{n-m}=\frac{(-1)^m}{m!}\det\begin{bmatrix} Tr(A) & m-1 & 0 & \cdots & \\ Tr(A^2) & Tr(A) & m-2 & \cdots & \\ \vdots & \vdots & & & \vdots \\ Tr(A^{m-1}) & Tr(A^{m-2}) & \cdots & \cdots & 1 \\ Tr(A^m) & Tr(A^{m-1}) & \cdots & \cdots & Tr(A) \end{bmatrix}.$$

5. The equality $rk(A)=Tr(A)$

Examples below show that the condition $rk(A)=Tr(A)$ is neither necessary nor sufficient for the matrix A to be idempotent.

The condition is not necessary.

Example. Take $E=4I_2$ over $\mathbb{Z}_6$. Then $E^2=E$, $Tr(E)=2$ and since $3E=0_2$, $rk(E)=0\neq2=Tr(E)$.

The condition is not sufficient, even if the necessary conditions $t(t-1)=2d$, $(t-2)d=0$, $d^2=d$ hold.

First note that $rk(A) = Tr(A)$ holds iff

(0) $rk(A) = Tr(A) = 0$. Since $t = 0$ it follows $2d = 0$. As $rk(A) = 0$, there exists a nonzero $r \in R$ such that $a = rb = rc = r(t-a) = 0$.

Take $A = 2E_{12}$ over any commutative ring of characteristics 4. Then $\det(A) = Tr(A) = 0$ and $2A = 0$ for $2 \neq 0$, so $rk(A) = 0$. The matrix is zerosquare, not idempotent.

(1) $rk(A) = Tr(A) = 1$. If $t = 1$ then from $(t-2)d = 0$ it follows $d = 0$ so $A = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$, is indeed idempotent (and as $d = 0$, $rk(A) = 1$).

(2) $rk(A) = Tr(A) = 2$. If $t = 2 = rk(A)$ then $\det(A) \neq 0$ is cancellable. From $d^2 = d$ it follows $d = 1$, so A is a unit.

However, A may not be (the only idempotent unit) I_2.

Indeed, from Cayley-Hamilton's theorem, we have $A^2 - 2A + I_2 = (A - I_2)^2 = 0_2$. Hence $A = I_2 + T$ with zerosquare T. So (unipotent) not necessarily idempotent.

Example. Over any ring, take $A = I_2 + E_{12} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \neq I_2 + 2E_{12} = A^2$. All three equalities, incl. $Tr(A) = rk(A) = 2$, hold.

Therefore, over arbitrary commutative rings the equality $rk(A) = Tr(A)$ is neither necessary nor sufficient for the matrix A to be idempotent

6. The equality $rk(A) + rk(I_2 - A) = 2$

Examples below show that the condition $rk(A) = Tr(A)$ is neither necessary nor sufficient for the matrix A to be idempotent.

The condition is not necessary.

Example. Take $E = 4I_2$ over $\mathbb{Z}_6$. Then $E^2 = E$, $Tr(E) = 2$ and since $3E = 0_2$, $rk(E) = 0 \neq 2 = Tr(E)$.

$I_2 - E = 3I_2$ is also (the complementary) idempotent and $2E = 0_2$ shows that $rk(I_2 - E) = 0$. The sum of both ranks is $= 0 \neq 2$.

The condition is not sufficient.

Due to the fact that the complementary of the complementary is the initial idempotent, it suffices to check (even *together with* the three necessary conditions on t and d) that

(i) $rk(A) = 0$, $rk(I_2 - A) = 2$ may not imply $A^2 = A$,

Example. Take $A = 2E_{12}$ over $\mathbb{Z}_6$. Then $rk(A) = 0$, $rk(I_2 - A) = 2$ (a unit), $Tr(A) = \det(A) = 0$, $t(t-1) = 2d$, $(t-2)d = 0$, $d^2 = d$ all hold, but A is not idempotent.

and

(ii) $rk(A) = 1 = rk(I_2 - A)$ may not imply $A^2 = A$.

Example. $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ was an example in Section 2: $rk(A) = 1$. Next, $I_2 - A = \begin{bmatrix} 5 & 0 \\ 0 & 4 \end{bmatrix}$ has a unit entry and zero divisor determinant. By Lemma 2.1, $rk(I_2 - A) = 1$.

As $d = 0$, the conditions $d^2 = d$, $(t-2)d = 0$ hold. Unfortunately, $t(t-1) = 2d$ fails. Clearly, A is not idempotent (this example is close but not complete).

1) Take $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ over $\mathbb{Z}_6$. Here $t = 0$, $d = 3$ so $d^2 = d$, $(t-2)d = 0$ and $t(t-1) = 2d$, hold. By Lemma 2.1, $rk(A) = 1$. Further, $I_2 - A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \end{bmatrix}$ has a unit entry and $\det(I_2 - A) = 4$, a zero divisor. Again, by Lemma 2.1, $rk(I_2 - A) = 1$. So $rk(A) + rk(I_2 - A) = 2$, but A is not idempotent.

However, $rk(A) = 1 \neq 0 = Tr(A)$.

2) Take $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$ over $\mathbb{Z}_6$, which is not idempotent ($A^2 = 3I_2 \neq A$). Here again $t = 0$, $d = 3$ so $d^2 = d$, $(t-2)d = 0$ and $t(t-1) = 2d$, hold. By Lemma 2.1 (i), $rk(A) = 0 = Tr(A)$.

However, by Lemma 2.1 (vii), $I_2 - A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ has rank $= 1$ and so $rk(A) + rk(I_2 - A) = 1 \neq 2$.

7. Final example

In closing, we provide an example of 2×2 matrix which satisfies all the above mentioned conditions (that is, the three conditions involving only trace and determinant and the two rank conditions) but is not idempotent.

Example. Take $A = 2I_2$ over $\mathbb{Z}_4$, which is nilpotent and so not idempotent. As $t = d = 0$, the three necessary conditions hold. As for the rank conditions:

(a) $rk(A) = 0 = Tr(A)$, because $Ann(I_1(A) = Ann(2R) = 2R \neq 0$.

(b) $I_2 - A = 3I_2$ has $\det(3I_2) = 1$ so is a unit. Hence $rk(I_2 - A) = 2$ and finally $rk(A) + rk(I_2 - A) = 2$.

References

[1] W. C. Brown *Matrices over commutative rings*. Marcel Dekker Inc., 1993.

[2] G. Călugăreanu 3×3 *idempotent matrices over some domains and a conjecture on nil-clean matrices*. Scientific Annals of "Al.I. Cuza" University, **68** (1) (2022), 91-106.