Units and nilpotent elements in corners

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Lemma 1 *Let A, B be subsets of a ring R with A a subgroup of $(R, +)$. If $a \in A$ then $a + (A \cap B) = A \cap (a + B)$.*

Proof. Since $a \in A$, both $a + (A \cap B) \subseteq A$ and $a + (A \cap B) \subseteq a + B$ are obvious. Conversely, if $a' \in A \cap (a + B)$ there is a $b \in B$ with $a' = a + b$ and so $b = a' - a \in A$. Hence $a' \in a + (A \cap B)$. ■

Lemma 2 *(i) Both $1 + N(R) \subseteq U(R)$ and $N(R) \subseteq 1 + U(R)$ hold for any ring R.*

(ii) $U(R) \subseteq 1 + N(R)$ and $1 + U(R) \subseteq N(R)$ are equivalent conditions.

Proof. Obvious. ■

Remark. Both conditions in (ii) define the UU rings.

Lemma 3 *Let $0 \neq e = e^2$ in a ring R. Then*

(a) $N(eRe) = (eRe) \cap N(R)$.

(b) $U(eRe) = (eRe) \cap (\overline{e} + U(R))$.

Proof. (a) Obvious.

(b) "$\subseteq$" If $u \in U(eRe)$ there is $v \in eRe$ with $uv = vu = e$. Since products of u or v with $\overline{e}$ are zero, $(u - \overline{e})(v - \overline{e}) = e + \overline{e} = 1 = (v - \overline{e})(u - \overline{e})$ and so $u - \overline{e} \in U(R)$ or $u \in \overline{e} + U(R)$.

"$\supseteq$" If $a = \overline{e} + u \in eRe$ with $u \in U(R)$, there exists $b \in R$ such that $(a - \overline{e})b = b(a - \overline{e}) = 1$. Left and right multiplication with e (respectively), give $ab = ba = e$ and so ebe is the inverse of a in eRe. ■

Theorem 4 *UU passes to corners.*

Proof. According to Lemma 2, we now just check: $e + U(eRe) = e + (eRe) \cap (\overline{e} + U(R)) \overset{Lemma 1}{=} (eRe) \cap (e + \overline{e} + U(R)) = (eRe) \cap (1 + U(R)) \overset{hypothesis}{=} (eRe) \cap \mathbf{N}(R) = \mathbf{N}(eRe)$. ■