# Equivalent definitions and generalizations for algebraic lattices

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**Abstract**

Formalizing some well-known routines in algebraic lattices, we give two new equivalent definitions for this class. A natural generalization of algebraic lattices is defined and discussed, called *compactic* lattice.

All lattices considered in the sequel will be **complete** lattices. We use the quotient sublattice notation from [1] for intervals.

## 1 General Case

We first formalize some well-known routines in algebraic lattices. This gives the following general frame.

Let $A$ be nonempty subset of a lattice $L$.

**Definitions.** The lattice $L$ is called *$A$-generated* if every element of $L$ is a join of elements from $A$ and *$A$-separated* if for every $a > b$ there is an element $x \in A$ with $x \leq a$ and $x \nleq b$. Moreover, denote by $a/A = \{x \in A | x \leq a\}$ and call a lattice *$A$-large* if $a \leq b$ whenever $\varnothing \neq a/A \subseteq b/0$. Finally, a lattice is *$A$-ctic*, if every nonzero element has a nonzero lower bound in $A$.

The special cases which could be of interest are $A = K = K(L)$ the set of all the compact elements of a lattice $L$ (notice that here $0 \in A$), $A = A(L)$ the set of all the atoms of $L$ (notice that here $0 \in A$) and $A = N(L)$ the set of all the Noetherian elements of $L$ (an element $a \in L$ is called *Noetherian* if the sublattice $a/0$ satisfies the ACC).

**Remark 1** *$A$-separated can (equivalently) be defined also as follows: if $b \ngeq a$, there is an element $x \in A$ with $x \leq a$ and $x \nleq b$.*

Indeed, since $b \ngeq a$ happens when $a > b$ or $a \| b$ (i.e., the elements are not comparable), this definition is apparently stronger. However, if $a \| b$ then $a > a \wedge b$, and by the first definition, there is an element $x \in A$ with $x \leq a$ but $x \nleq a \wedge b$. Since $x \leq b$ would imply $x \leq a \wedge b$, we have $x \nleq b$, as desired.

Note that *a lattice is $A$-generated if and only if for every $a \in L$, $a = \bigvee a/A$* (i.e., $= \bigvee\{x | x \in a/A\}$).

**Theorem 2** *The following conditions are equivalent for a lattice $L$:*

*(i) $L$ is $A$-separated;*

*(ii) L is A-generated;*
*(iii) L is A-large.*

**Proof.** (i) $\Longrightarrow$ (ii) Suppose $L$ is not $A$-generated. Using the equivalent definition mentioned above, there exists an element $b \in L$ such that $b > \bigvee b/A$. Denoting $a = \bigvee b/A$, clearly $b/A \subseteq a/A$ and so $a < b$ are not separated (and so $L$ is not $A$-separated). It is easy to obtain from this (i) $\Longrightarrow$ (iii).

(ii) $\Longrightarrow$ (iii)

Further, if the lattice is $A$-generated, it is also $A$-large. Indeed, since $a/A \subseteq b/0$ implies $a/A \subseteq b/A$, we obtain $\bigvee a/A \leq \bigvee b/A$ and the previous remark gives $a \leq b$.

(iii) $\Longrightarrow$ (i) If $a > b$, suppose there is no $x \in a/A$ which satisfies $x \nleq b$ (i.e., $L$ is not $A$-separated). Then $a/A \subseteq b/0$ and since $a \nleq b$ the lattice is not $A$-large. ■

**Proposition 3** *Every A-generated lattice is A-ctic.*

**Proof.** Indeed, if $0 \notin A$, let $0 \neq a \in L$. Since $a = \bigvee a/A$ and $\bigvee \varnothing = 0$, we derive $a/A \neq \varnothing$ and $L$ is $A$-ctic. If $0 \in A$ then we have to compare $a = \bigvee a/A$ and $\bigvee\{0\} = 0$ in order to obtain $\{0\} \subsetneqq a/A$, and again $L$ is $A$-ctic. ■

**Remark 4** *If a lattice is A-separated (or equivalently A-generated), then A contains the atoms of L.*

Indeed, atoms $a > 0$ are separated from 0 (if and) only if $a \in A$.

*Summarizing*

$A$-generated $\Longleftrightarrow$ $A$-separated $\Longleftrightarrow$ $A$-large $\Longrightarrow$ $A$-ctic.

## 2 Special case $A = K(L)$

In this study we concentrate on the first special case, that is $A = K$, the compact elements of a lattice $L$.

We shall use the terms *compactly-separated*, *compactly-large* and *compactic* (for $K$-ctic). Actually, in compactly generated lattices, (very often) *this is the way someone checks an inequality* $a \leq b$: it suffices to check $c \leq b$ for all compact elements $c \leq a$ (that is, the compactly-large definition).

From the previous section we have at once

**Corollary 5** *The following conditions are equivalent for a lattice L:*
*(a) L is compactly generated*
*(b) L is compactly-separated*
*(b) L is compactly-large.*

For the sake of completeness, recall that in the literature (see [1]), both proofs for

**Proposition 6** *(i) Every compactly-generated lattice is weakly atomic, and*
*(ii) Every compactly-generated lattice is upper continuous,*

actually rely on the compactly-separated property. Examples are customarily given in order to show that these (last two) inclusions are proper (e.g., see [1]).

**Proposition 7** *All these classes are included in the atom-compact class (i.e., every atom is compact).*

**Proof.** Compactic lattices: obvious.

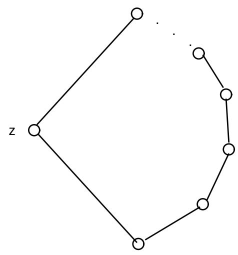
Upper continuous lattices: let $a \in L$ be an atom in an upper continuous lattice $L$. If $a \leq \bigvee D$ for an upper directed subset $D \subseteq L$ observe that for each $d \in D$, $a \wedge d = a$ or $a \wedge d = 0$. Hence $a \nleq d$ for each $d \in D$ is not possible: we would have $a = a \wedge (\bigvee D) = \bigvee (a \wedge d) = \bigvee 0 = 0$. ■

Weakly atomic does not generally imply atom-compact. See the next example.

*Summarizing*

finite $\Longrightarrow$ Noetherian $\Longrightarrow$ [compactly generated $\Longleftrightarrow$ compactly-separated $\Longleftrightarrow$ compactly-large] $\left\{ \begin{array}{c} \nearrow \text{ compactic } \searrow \\ \searrow \text{ upper continuous } \nearrow \end{array} \right.$ atom-compact.

An *example* of lattice which is not atom-compact is given in [3] (Figure 3.1 (b), p. 28)




the element $z$ is not compact, and in this case nor join of compact elements ($z$ is an atom which is not compact). So none of the properties in our first chart, but the lattice is Artinian and so (strongly) atomic and weakly atomic.

**Remark 8** *If* $0$ *is the only compact of a lattice* $L$*, this lattice is not compactic, nor compactly-generated.*

Indeed, for every $x, y \in L$, $x/K = \{0\} = y/0$ would imply $x \leq y$ and $y \leq x$.

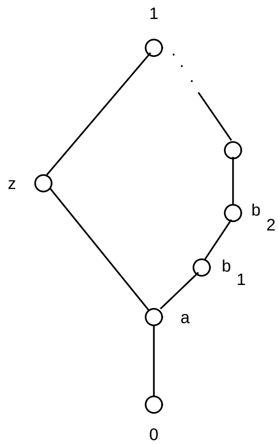
Such examples abound:

1) $([0,1], \leq)$, i.e., the interval of real numbers with the usual order.

2) Jeffrey Leon's example ([1], p. 16): invented in order to show that an upper continuous weakly atomic lattice is not always compactly generated.

3) $L = \{0\} \cup \{2^r 3^s | r, s \in \mathbf{N}\}$ be ordered by divisibility ([2], **Ex. 7.2**, p. 169).

In order to legitimate the compactic class of lattices, we supply *an example of compactic lattice, which is not compactly generated.*




Indeed, the element $z$ is not compact (the cover $z \leq \bigvee_n b_n$ has no finite subcover), nor join of compact elements (hence the lattice is not compactly generated). However, every (nonzero) element has a compact lower bound ($a$, $b_1$, $b_2$, ... are compact and $z$ and 1 have compact lower bounds), and so the lattice is compactic.

More, this lattice:

it is not upper continuous - $z = z \wedge 1 = z \wedge (\bigvee b_n) \neq \bigvee (z \wedge b_n) = \bigvee a = a$;

it is not compactly large - $z/K = \{a\} \subseteq b_1/0$ but $z \leq b_1$ fails;

it is atomic, and atom-compact, but not atom generated

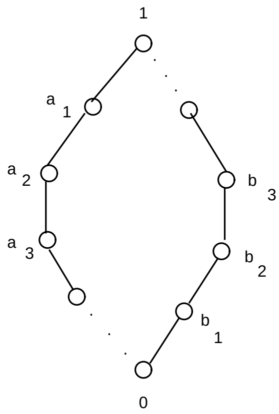
it is not Noetherian, but it is Artinian and so strongly atomic.

The example above also shows that *compactic* $\Longrightarrow$ *upper continuos, fails.*

The converse also fails: *an example of upper continuous lattice which is not compactic* is the Jeffrey Leon example mentioned above.

Finally, in order to show that all inclusions are proper, here are some *examples of atom-compact lattice which is not compactic*

First of all, if 0 is the only compact element, the lattice is trivially atom-compact if it has no atoms! And $([0, 1], \leq)$, i.e., the interval of real numbers with the usual order, is such an example. Less trivial (but somehow similar) we have




None of $a_n$ is compact and so has no nonzero compact lower bound. The only atom $b_1$ is compact.

# 3 Theoretical example ?

For an arbitrary lattice $L$, both the lattice of all the *ideals* $I(L)$ and the lattice of all *congruences* $\mathrm{Con}(L)$ are compactly-generated.

The lattice of all the *convex sublattices* $CS(L)$ is also atomic and compactly generated.

*It would be interesting to find a theoretical example of lattice associated with an (arbitrary) lattice which is compactic, but generally not compactly-generated.*

...

**Final comment.** Since compact elements in an arbitrary sublattice cannot be related to the compact elements in the whole lattice, as compactly generated lattices do, this new class does not form a (quasi)variety. Actually, quotient of complete lattices may be non-complete, so since compact element definition needs completeness, there is *no hope for a variety. Nor quasivariety*, because of sublattices.

# References

[1] Crawley P., Dilworth R. *Algebraic Theory of Lattices*. Prentice Hall, Englewood Cliffs, N. J., 1973

[2] Davey B.A., Priestley H.A. *Introduction to Lattices and Order*. Cambridge University Press, Fourth Printing 2008.

[3] Nation J.B. *Notes on Lattice Theory*. University of Hawaii.