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ORIGINAL ARTICLE

Central stable range one for elements and rings

Grigore Călugăreanu[1]

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Abstract

An element a of a ring R is said to have *left central stable range one* if whenever $Ra + Rb = R$ for any $b \in R$, there exists $z \in Z(R)$ (the center of R) such that $a + zb$ is a unit of R. A ring R has *left central stable range one* if all its elements have it. In this work, elements and rings with left central stable range one are studied. The condition is left-right symmetric for such rings.

Keywords Central stable range one element · Matrix ring · Triangular matrix ring

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1 Introduction

All rings considered in this work are assumed to be associative with an identity element.

For any subset S of a ring R, an element a is said to have *left S-stable range 1* if, whenever $Ra + Rb = R$ (for any b), there exists $s \in S$ such that $a + sb$ is a unit. We denote the set of all left S-stable range 1 elements of R by $sr_S(R)$. When S is R itself, $U(R)$ (the units of R), $Id(R)$ (the idempotents of R), or $N(R)$ (the nilpotents of R), this corresponds to the notions of left stable range 1 of R (due to Bass [1]), left unit stable range 1 of R (due to Goodearl and Menal [8]), left idempotent stable range 1 of R (due to Chen [3]), and recently, left nilpotent stable range 1 of R (due to Zhou [19]). To simplify notation, $sr1$ will be used to abbreviate left stable range one. The corresponding elements are customarily denoted $sr(R)$, the left sr1 elements, $usr(R)$, the left unit sr1 elements, $isr(R)$, the left idempotent sr1 elements and $nsr(R)$, the nilpotent sr1 elements. Clearly, left S-stable range 1 elements have stable range 1.

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Grigore Călugăreanu
calu@math.ubbcluj.ro

1 Department of Mathematics, Babeş-Bolyai University, 400084 Cluj-Napoca, Romania

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This paper discusses the special case $S = Z(R)$, the center of R. An element a is said to have *left central stable range 1* if, whenever $Ra + Rb = R$ (for any b), there exists $z \in Z(R)$ such that $a + zb$ is a unit. A ring has *left central stable range 1* if all its elements have this property. *Right central stable range 1* elements and rings are defined symmetrically, considering right principal ideals instead of left principal ideals.

In what follows, central sr1 will be used to designate left central stable range 1 (unless otherwise stated). More precisely, $csr(a) = 1$ means a has left central sr1, and $csr(R)$ denotes the set of all the left central sr1 elements of R.

It is easy to see that if $sr_S(a) = 1$, then a is the sum of a unit and an element in S (take $b = -1$ in $a + yb$).

Then, for various choices of S:

- For $S = U(R)$, $usr(R) \subseteq U(R) + U(R) =$ {2-good elements},
- For $S = Id(R)$, $isr(R) \subseteq U(R) + Id(R) =$ {clean elements},
- For $S = N(R)$, $nsr(R) \subseteq U(R) + N(R) =$ {fine elements},
- For $S = Z(R)$, $csr(R) \subseteq U(R) + Z(R)$.

Rings for which $R = U(R) + Z(R)$ were studied in [11] (called *CU* rings).

Our main results are as follows:

- in any ring R with $S \subseteq R$, we prove the *left-right symmetry of S-stable range 1*, whenever $1 - S \subseteq S$,
- in any ring R, we show 0 has central sr1 in a ring R if and only if R is Dedekind finite,
- we relate $csr(R)$ with $csr(R/I)$ for ideals I of R,
- we provide a sufficient condition for the product of two elements with central sr1 to have central sr1,
- in any matrix ring, we show that idempotent matrices $\sum_{i=1}^{k} E_{ii}$, for any $1 \leq k \leq n-1$, do not have central sr1 in R,
- we determine $csr(\mathbb{M}_2(\mathbb{Z}))$: only zero and invertible integral matrices have central sr1,
- we identify the central sr1 elements in products of rings, rings of formal power series, trivial extensions of a ring R by an R-module M, formal triangular matrix rings and rings of triangular matrices, respectively.

The structure of the paper is as follows: Section 2 provides general definitions and results related to S-stable range 1, which are specialized for $S = Z(R)$ in Section 3.

The left-right symmetry of elements with S-stable range 1, in relationship with the so-called Super Jacobson's lemma, is discussed in Section 2. The left-right symmetry of rings with S-stable range 1 is proved whenever $1 - S \subseteq S$.

Idempotents, units, and more generally, unit-regular elements, all have sr1. Examples and results on central sr1 for such elements in arbitrary (unital) rings, including nilpotents and idempotent matrices, are discussed in Section 3. The behavior of central sr1 towards some well-known Ring Theory constructions is described in Section 4.

In closing, an open question is stated.

As customarily, for any positive integer $n \geq 2$, E_{ij} denotes the $n \times n$ matrix with all entries zero except for the (i, j)-entry, which equals 1. A diagonal square matrix is called *scalar* if all diagonal entries are equal. A ring R is called a *GCD* ring if the greatest common divisor of every pair of elements of R exists.

2 *S*-stable range one

In this section we gather general definitions and results related to stable range one, relative to a subset S of a ring R. As already mentioned, an element a in R is said to have *left S-stable range 1* if, whenever $Ra + Rb = R$ (for any $b \in R$), there exists $s \in S$ such that $a + sb \in U(R)$. We write $sr_S(a) = 1$, meaning that a has left S-stable range 1.

Proposition 2.1 *Consider the following conditions for an element $a \in R$:*

(1) $sr_S(a) = 1$,
(2) *for every $x, y, b \in R$, if $xa + yb = 1$ then there exists $s \in S$ such that $a + sb \in U(R)$,*
(3) *for every $x, b \in R$, if $xa + b = 1$ then there exists $s \in S$ such that $a + sb \in U(R)$,*
(4) *for every $x \in R$, there exists $s \in S$ such that $a + s(1 - xa) \in U(R)$.*

Then (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4), but (3) $\Rightarrow$ (2) may fail.

Proof As $Ra + Rb = R$ is equivalent to $1 \in Ra + Rb$, it follows that (1) $\Leftrightarrow$ (2). Eliminating b from (3) (i.e., $b = 1 - xa$), it follows that (3) $\Leftrightarrow$ (4). Clearly, taking $y = 1$, (2) $\Rightarrow$ (3). However, in [19] it is shown that, for $S = N(R)$, (3) $\not\Rightarrow$ (2) and this shows that (1) $\Rightarrow$ (4) but (4) $\Rightarrow$ (1) may fail. □

Remarks 1) A sufficient condition which makes (2) and (3) equivalent is $SR \subseteq S$, that is, S is a right ideal in the multiplicative monoid $(R, \cdot)$. It is readily seen that none of the subsets $U(R)$, $Id(R)$, $N(R)$, $reg(R)$ (the von Neumann regular elements of R), $ureg(R)$ (the unit-regular elements of R), $sreg(R)$ (the strongly regular elements of R) and even the subring $Z(R)$, is such a right ideal.

However, for $S = U(R)$, that is, for unit-sr1 elements, without any other hypothesis, (2) and (3) were proved to be equivalent (see Lemma 1.1, [8]).

2) The denial of (4) can be used when showing that an element has not central sr1.

A recent result of Khurana and Lam shows that an element a in a ring has left stable range 1 iff it has right stable range 1 (see [10], Theorem 3.1). The proof uses the so-called *Super Jacobson's lemma*, which states that for any three elements a, b, x of a ring R, $a + b - axb \in U(R)$ iff $a + b - bxa \in U(R)$ (see

[10], Lemma 3.2). In [10] it is proved that this holds replacing $U(R)$ by $reg(R)$ or $ureg(R)$ but fails replacing by $sreg(R)$.

Below we show that this also fails replacing $U(R)$ by $Z(R)$, that is, we give an example for $a+b-axb \in Z(R)$ but $a+b-bxa \notin Z(R)$.

Example In the ring $R = \mathbb{M}_2(\mathbb{Z})$, let $A = E_{11}$, $B = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$ and the unit $X = -E_{12} - E_{21}$. Then $A + B - AXB = 2I_2 \in Z(\mathbb{M}_2(\mathbb{Z}))$ (it is a scalar matrix). However, $A + B - BXA = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix} \notin Z(\mathbb{M}_2(\mathbb{Z}))$ (it is not a scalar matrix).

For any subset S of a ring R, the Super Jacobson's lemma has the following consequence

Proposition 2.2 *Let $a, x \in R$ and $s \in S$. Then $a + s(1 - xa) \in U(R)$ iff $a + (1 - ax)s \in U(R)$.*

Rephrasing, the "left" condition (4) (see above Proposition 2.1) is equivalent to the "right" condition (4). Clearly this holds also for $S = Z(R)$.

However, since the condition (4) is not equivalent to (1), *the left-right symmetry* of S-stable range one *of elements* for all the subsets $S = Id(R)$, $N(R)$ and even for the *subring* $Z(R)$, cannot be proved using the previous proposition.

However, *for rings* we can prove the *left-right symmetry of S- stable range 1*, just adding the condition $1 - S \subseteq S$.

First we just recall from [3] the following notations and results (Theorems 4, 5 and Corollary 6).

Over any (unital) ring R, denote some special 2×2 units by $B_{12}(x) := I_2 + xE_{12}$, $B_{21}(y) := I_2 + yE_{21}$ and $[u, v] := \mathrm{diag}(u, v) = uE_{11} + vE_{22}$ for $x, y \in R$ and $u, v \in U(R)$.

Theorem 2.3 *The following are equivalent*:

(a) *R has idempotent stable range* 1.
(b) *For any $A \in GL_2(R)$, there is an idempotent $e \in R$ such that $A = [*, *]B_{21}(*)B_{12}(*)B_{21}(-e)$.*
(c) *For any $x, y \in R$, there is an idempotent $e \in R$ such that $xy - xe + 1 \in U(R)$.*

Corollary 2.4 *A ring has right idempotent $sr1$ iff it has left idempotent $sr1$.*

A careful reading of the proofs shows that (a) and (b) are equivalent replacing $Id(R)$ by *any* subset S of R, (a) $\Rightarrow$ (c) holds for *any* subset S of R and if $1 - S \subseteq S$ then also (c) $\Rightarrow$ (a) holds. Therefore the corollary holds replacing $Id(R)$ by any subset S of R, if we only add the condition $1 - S \subseteq S$. Therefore

Theorem 2.5 *Let S be a subset of a ring R and suppose $S \subseteq 1 - S$. Then R has left S-stable range one iff it has right S-stable range one.*

Finally, a general definition and some straightforward observations, to be used in the next section.

Definition Let S be a subset of a ring R. The ring R is called *S-central* if $S \subseteq Z(R)$, and *S-commuting* if every two elements of S commute.

Lemma 2.6 *Let S and T be subsets of a ring R.*

(a) *If R is S-central and T-central then it is also $(S+T)$-central and ST-central.*
(b) *If $S \subseteq T \subseteq R$ and R is T-central then R is also S -central.*
(c) *If R is S-central then R is also S-commuting.*
(d) *For $S = Id(R)$, note that R is $Id(R)$-central iff it is $Id(R)$ -commuting. Such rings were called Abelian.*

Proof (d) One way is covered by (c). As for the converse, let $e = e^2 \in R$ be an idempotent. Then $e \in Z(R)$ iff e commutes with all the idempotents or R which are isomorphic to e (see **22.3.A** in [12]). □

3 Central stable range one

In this section we specialize the results from the previous section to $S = Z(R)$, the center of a ring R. First some

3.1 Examples

(1) Since zero is central in any ring, *units have left and right central sr1*.
(2) In any commutative ring, an element has (left or right) central sr1 iff it has sr1.
(3) A *nonunit* example with $csr(a) = 1$ is any nontrivial idempotent in $\mathbb{Z}_n$ for any $n \geq 2$, say 3 in $\mathbb{Z}_6$ (this ring is also unit-regular, so $(c)sr(\mathbb{Z}_6) = 1$).

It is easy to provide *classes of central sr1 rings* by mixing classes which were already studied.

(i) The unit sr1 rings which are unit-central (discussed in [8] and [9]).
(ii) The idempotent sr1 rings which are Abelian (discussed in [3] and [17]).
(iii) The nilpotent sr1 rings which are nilpotent-central (discussed in [19] and [15]).

Among these, the *local rings*, and for *Abelian* rings, the *clean* rings or equivalently the exchange rings (i.e., a special class of strongly clean rings called *topologically-Boolean* in [5]; see also [4]).

Or else, also for Abelian rings, the weakly P-exchange rings. A ring R was called *weakly P-exchange* if every right R-module has finite exchange property (cf. [14]). It is well known that regular rings, right perfect rings and weakly right perfect rings are all weakly P-exchange. In particular, it follows that Abelian regular, or equivalently, *strongly regular rings have central sr1*.

Central sr1 rings need not be semilocal. Indeed, a direct product of an infinite family of fields has central sr1 (see Proposition 4.2) but is not semilocal.

Semilocal rings have sr1, but may not have central sr1. Indeed, let D be a division ring and $R = \mathbb{M}_n(D)$. Then R is left Artinian and so semilocal. However (see Corollary 3.7), $csr(R) = \{0\} \cup U(R) \neq R$.

Clearly, also *commutative* unit (or idempotent or nilpotent) sr1 rings are central sr1.

From [18], we know that *commutative exchange rings and exchange PI-rings have stable range 1*.

Hence, commutative exchange rings have also central sr1.

Finally, if R is unit-central then $N(R) \subseteq Z(R)$. Hence unit-central nilpotent sr1 rings are also central sr1.

3.2 Central sr1 elements

In [19] it was noticed that for $S = N(R)$, $sr_{N(R)}(0) \neq 1$. As for $S = Z(R)$ we have the following result.

Proposition 3.1 *0 has central sr1 in a ring R iff R is Dedekind finite.*

Proof By (2) in the Proposition 2.1, $csr(0) = 1$ iff $cb = 1$ implies the existence of some $z \in Z(R)$ such that $zb \in U(R)$.

To show the condition is necessary, notice that as z is central, both z, b must be units. Hence, 0 may have central sr1 only if the ring is Dedekind finite (one-sided invertible elements are units). The condition is also sufficient. Indeed, if R is Dedekind finite and $cb = 1$ then b is a unit and we can choose $z = 1 \in Z(R)$. □

Corollary 3.2 *Division rings have central sr1.*

There are many classes of rings for which 0 has central sr1.

A ring is *unit-central* if $U(R) \subseteq Z(R)$ and *nilpotent-central* if $N(R) \subseteq Z(R)$. In all the classes of rings in the following diagram, 0 has central sr1.

$$\begin{array}{ccccccc}
 & & \text{commuting.units} & & & & \\
 & \nearrow & & \searrow & & & \\
\text{unit.central} & \rightarrow & \text{nilp.central} & \rightarrow & \text{uni} & \rightarrow & \text{commuting.nilp} \\
 & & \downarrow & & & & \downarrow \\
 & & \text{Abelian} & & \longrightarrow & & \text{DF}
\end{array}$$

Here a ring was called *uni* (see [2]) if all units commute with all nilpotents and DF stands for Dedekind-finite.

Lemma 3.3

(1) *If* $csr(a) = 1$ *then* $a \in U(R) + Z(R)$.
(2) $csr1$ *elements are invariant under equivalences.*
(3) *If* $a \in crs(R)$ *and* $I \subseteq J(R)$ *is an ideal of* R *then* $a + I \in csr(R/I)$. *The converse holds if central elements lift modulo* I. *In particular*, $csr(R) = 1$ *implies* $csr(R/J(R)) = 1$ *and the converse holds if central elements lift modulo* $J(R)$.

Proof

(1) Since $Ra + R(-1) = R$, there is $z \in Z(R)$ such that $a - z \in U(R)$. Hence $a \in U(R) + Z(R)$.
(2) Suppose $R(uav) + Rb = R$ for some units u, v. Then $R = Rv^{-1} = Rua + Rbv^{-1} = Ra + Ru^{-1}bv^{-1}$. By hypothesis, there is $z \in Z(R)$ with $a + zu^{-1}bv^{-1} \in U(R)$. By left and right multiplication $u(a + zu^{-1}bv^{-1})v = uav + uzu^{-1}b = uav + zb \in U(R)$.
(3) Denote $\overline{R} = R/I$ and $\overline{a} = a + I$. Assume that $\overline{R}\overline{a} + \overline{Rb} = \overline{R}$. Then $Ra + Rb + I = R$, so $Ra + Rb = R$ as $I \subseteq J(R)$. Hence, $a + zb \in U(R)$ for some $z \in Z(R)$. It follows that $\overline{a} + \overline{z}\overline{b} \in U(\overline{R})$ with $\overline{z} \in Z(\overline{R})$.

For the converse, assume that $Ra + Rb = R$. Then $\overline{R}\overline{a} + \overline{Rb} = \overline{R}$. So $\overline{a} + \overline{z}\overline{b} \in U(\overline{R})$ for some $\overline{z} \in Z(\overline{R})$. By hypothesis we can assume $z \in Z(R)$ and so $a + zb \in U(R)$, since units lift mod I, if $I \subseteq J(R)$. □

If central elements do not lift modulo some ideal I, then the converse may fail.

Examples (1) First an example of central element which does not lift modulo the Jacobson radical.

Consider $R = \mathbb{T}_2(F)$, the upper triangular matrices over a field F and $J(R) = \{aE_{12} : a \in F\}$, the strictly upper triangular matrices (i.e., the matrices with zero diagonal entries). Central elements do not lift modulo $J(R)$.

Indeed, recall that $R/J(R) \cong \mathrm{diag}_2(F)$ is the commutative ring of the diagonal 2×2 matrices over F. Since $Z(\mathrm{diag}_2(F) = \mathrm{diag}_2(F)$, any not scalar diagonal does not lift to a scalar matrix in R (the center $Z(\mathbb{T}_2(R)) = \{aI_2 : a \in F\}$).

However $csr(R) = 1$ according to Corollary 4.4, (4).

(2) Secondly, an example of a ring R for which central elements do not lift modulo some ideal I and an element $a \in R$ such that $csr(a + I) = 1$ but $csr(a) \neq 1$.

Now take $R = \mathbb{T}_2(S)$, the upper triangular matrices over a ring S which is not DF, take the ideal $I = \begin{bmatrix} S & S \\ 0 & 0 \end{bmatrix}$ and take $a = E_{22} \in R$. According to Corollary 4.4, (4), $csr(E_{22}) \neq 1$ (as, by Proposition 3.1, 0 has not csr1). However $E_{22} + I$ is the identity in R/I so $csr(E_{22} + I) = 1$.

For products of central stable range 1 elements we can prove the following result.

Proposition 3.4 *Let $a, a' \in R$. If $csr(a) = csr(a') = 1$ and $a \in Z(R)$ then $csr(aa') = 1$.*

Proof We first show that *for any element a of a ring R, $csr(a) = 1$ iff for any $b \in R$, $Ra + Rb = R$ implies that $csr(a + zb) = 1$ for some $z \in Z(R)$.*

$\Rightarrow$ is obvious, as we can take $z = 0$. As for $\Leftarrow$, suppose $Ra + Rb = R$. Then there exists $z \in Z(R)$ such that for $a' = a + zb$ we have $csr(a') = 1$. It follows that $Ra' + Rb \supseteq Ra + Rb = R$ implies that there exists $z_0 \in Z(R$ such that $a' + z_0 b \in U(R)$. As $a' + z_0 b = a + (z + z_0)b$ this shows $csr(a) = 1$.

Secondly, for the proof of the proposition, suppose $Raa' + Rb = R$. Then $Ra' + Rb = R$ and so there exists $z \in Z(R)$ such that $u := a' + zb \in U(R)$. By left multiplication, $aa' + azb = au$ with $csr(au) = 1$ (equivalent to a). Since $a \in Z(R)$, according to the first part, this shows that $sr(aa') = 1$. □

3.3 Central sr1 matrices

In order to give examples of *idempotents* with central sr1, we start with the ring R of upper triangular matrices with constant diagonal over a division ring k. It is well-known that R is a local (noncommutative) ring with $N(R) = J(R)$, the matrices with zero diagonal. As already mentioned in the examples subsection it follows that

Proposition 3.5 *The ring of upper triangular matrices with constant diagonal over any division ring has central sr1.*

Proof The claim follows using also Proposition 3.1. Indeed, local rings are Dedekind finite. □

In matrix rings over unital rings, left (or right) *central sr1 matrices are scarce*, due to the special form of the central matrices. Indeed, it is well-known that $Z(\mathbb{M}_n(R)) = \{zI_n : z \in Z(R)\}$, that is, the center consists of the so called *scalar* matrices.

First we prove the following result.

Theorem 3.6 *Let $R = \mathbb{M}_n(S)$ for some (unital) ring S. The idempotent $n \times n$ matrices $A = \sum_{i=1}^{k} E_{ii}$, for any $1 \leq k \leq n-1$, have not central sr1 in R.*

Proof We provide $B \in R$ such that $RA + RB = R$ but $A + ZB$ is not a unit, for any $Z \in Z(R)$.

Consider $B = E_{k,k+1} + ... + E_{n-1,n}$, a lower part of the superdiagonal. Left multiplying B by $U = I_n + E_{k+1,k} + ... + E_{n,n-1}$ (equivalently, adding the $n-1$ row to the nth, the $n-2$ row to the $n-1$, ..., the k th row to the $k+1$ row) we get $A + UB = I_n + B$ and so $RA + RB = R$. Finally, for any $z \in D$ and $Z = zI_n \in Z(R)$, the matrix $A + ZB = A + zB$ is not a unit (in R). Indeed, it has the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & z & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & z & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & z & 0 & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & z \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

and any (upper triangular or not) matrix with a zero row is not invertible in R. □

Corollary 3.7 *If* $R = \mathbb{M}_n(D)$ *where* D *is a division ring and* $n \geq 1$, *then* $csr(R) = U(R) \cup \{0\}$.

Proof There is nothing to prove if $n = 1$, so we assume $n \geq 2$. Let $0 \neq A \in R$ and assume that A is not a unit. Then A is equivalent to $\sum_{i=1}^{k} E_{ii}$ where $1 \leq k \leq n-1$. Hence, by Lemma 3.3, (2) we can assume that $A = \sum_{i=1}^{k} E_{ii}$ and the result follows from the previous theorem. □

Since division rings have central sr1, it follows that matrix rings over csr1 rings may not be csr1. Hence, the central sr1 condition is not Morita invariant. However, it passes to corners (see next section).

Corollary 3.8 *Over any GCD integral domain, the nontrivial idempotent* 2×2 *matrices have not left (nor right) central sr*1.

Proof It suffices to recall (e.g., see Proposition 18, [7]) that every nontrivial idempotent 2×2 matrix over a GCD integral domain R is similar to E_{11}. □

Following Steger [13], we say that a ring R is an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings. Then

Corollary 3.9 *Over any connected ID ring, the nontrivial idempotent matrices have not left (nor right) central sr*1.

Finally, a consequence for integral matrices.

Corollary 3.10 *Let* $R = \mathbb{M}_2(\mathbb{Z})$. *Then* $csr(R) = 0_2 \cup U(R) \subsetneq U(R) + Z(R)$.

Proof Since matrix rings over commutative rings are Dedekind finite, $0_2 \in csr(R)$. As $U(R) \subseteq csr(R)$ generally holds, we start with $0_2 \neq A \in csr(R)$ and show that $A \in U(R)$. As $\mathbb{Z}$ is an elementary divisor domain, A is equivalent to a diagonal matrix $\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ with $a_1 \geq 1$, $a_2 \geq 0$ and $a_1 \mid a_2$. By Lemma 3.3, (2), we may assume $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$. By Theorem 11, [6], a matrix $A \in R \backslash U(R)$ has sr1 iff $\det(A) = 0$, so A is a unit or $\det(A) = 0$. We show that the later leads to a contradiction. If $\det(A) = 0$ we may assume $A = aE_{11}$ with $a \geq 1$ ($crs(a) = 1$ iff $crs(-a) = 1$ in any ring).

Let $B = \begin{bmatrix} 1-a & 0 \\ 0 & 1 \end{bmatrix}$. Then $Ra + Rb = R$ and there exists $Z = nI_2 \in Z(R)$ such that $U := A + ZB \in U(R)$. Thus $U = \begin{bmatrix} a + n(1-a) & 0 \\ 0 & n \end{bmatrix}$ and $\det(U) = n[a + n(1-a)]$ whence it follows that $n = 1$ or $n = -1$. In the first case, $U = A + B = E_{22}$ is not a unit and, in the second case $a = 1$ (as $a \neq 0$), that is, $A = E_{11}$, case which was covered by the previous theorem.

To show that the right inclusion may be proper, it is easy provide a 2×2 integral matrix in $U(R) + Z(R)$ which has no sr1 (and so neither crs1). Indeed, $I_2 + I_2 = 2I_2$ has not sr1 (see again Theorem 11, [6]). □

Remark In Proposition 2.1, we saw that the condition (4), that is, for every $x \in R$, there exists $z \in Z(R)$ such that $a + z(1 - xa) \in U(R)$, may be more general than $crs(a) = 1$.

However, for $R := \mathbb{M}_2(\mathbb{Z})$, (with entries bounded by 50) the *computer* did not find $A \neq 0_2$, $\det(A) = 0$ such that for all $X \in \mathbb{M}_2(\mathbb{Z})$ there exists $n \in \mathbb{Z}$ (depending on A and X) such that

$$\det[A + n(I_2 - XA)] = \pm 1.$$

So in this case (note that $Z = nI_2$) it seems that (1) and (4) are equivalent.

Question. Can we prove that for 2×2 integral matrices (1) and (4) are equivalent ?

As the reader has already noticed, for many (DF) rings, $csr(R) = U(R) \cup \{0\}$. Another well-known example is any polynomial ring $S[x]$ over a commutative domain S. The set of (central) stable range one elements is just $\{0\} \cup U(S)$.

4 Ring theoretic constructions

As an immediate consequence of Theorem 2.5, we record the left-right symmetry for rings of the central sr1, that is

Corollary 4.1 *A ring has left central sr1 iff it has right central sr1.*

Observe that the theorem does not apply for $S \in \{U(R), N(R)\}$.

As $\mathbb{Z}$ has not sr1 and $\mathbb{Q}$ has sr1, it follows that *subrings* of rings which have (central) sr1 may not have (central) sr1.

Proposition 4.2 *Let $R = \prod_{i \geq 1} R_i$ be a direct product of rings, and $\alpha = (a_i) \in R$. Then $\alpha \in csr(R)$ iff $a_i \in csr(R_i)$ for all i.*

Proof Let $\alpha = (a_i) \in csr(R)$ and assume that $R_1 a_1 + R_1 b_1 = R_1$. With $\beta = (b_1, 1, 1, ...) \in R$, $R\alpha + R\beta = R$. So $\alpha + \gamma\beta \in U(R)$ for some $\gamma = (z_i) \in Z(R)$. It follows that $a_1 + z_1 b_1 \in U(R_1)$ with $z_1 \in Z(R_1)$. So $a_1 \in csr(R_1)$. Similarly, $a_i \in csr(R_i)$ for each $i \geq 2$.

Conversely, let $\alpha = (a_i) \in R$ with $a_i \in csr(R_i)$ for all i, and assume that $R\alpha + R\beta = R$ where $\beta = (b_i) \in R$. Then, for each i, $R_i a_i + R_i b_i = R_i$, so there exists $z_i \in Z(R_i)$ such that $a_i + z_i b_i \in U(R_i)$. It follows that $\alpha + \gamma\beta \in U(R)$, where $\gamma = (z_i) \in Z(R)$. □

Remark Subdirect products of central sr1 rings may not be central sr1. For example $\mathbb{Z}$ is a subdirect product of the fields $\mathbb{F}_p$ $(p = 2, 3, 5, ...)$, but of course $\mathbb{Z}$ has not (central) sr1.

Proposition 4.3 *Let $R = S + J$ where S is a unital subring of R and $J \subseteq J(R)$ is an ideal such that $S \cap J = 0$. Let $a = s + j \in R$ with $s \in S$ and $j \in J$. Then $a \in csr(R)$ iff $s \in csr(S)$.*

Proof First note that the assumptions imply that $1 \in S$. Let $a = s + j \in csr(R)$ where $s \in S$ and $j \in J$ and assume that $Ss + Ss' = S$. Then $xs + ys' = 1$ where $x, y \in S$. Thus $xa + ys = (xs + ys') + xj = 1 + xj \in U(R)$. So $Ra + Rs' = R$, and hence $a + zs' \in U(R)$ where $z \in Z(R)$. Write $z = r_1 + j_1$ where $r_1 \in S$ and $j_1 \in J$.

Then $r_1 \in Z(S)$. Indeed, if σ is arbitrary in S then $z\sigma = \sigma z$, so $(r_1 + j_1)\sigma = \sigma(r_1 + j_1)$. Hence $r_1\sigma - \sigma r_1 = \sigma j_1 - j_1\sigma \in S \cap J = 0$ and $s + r_1 s' = (a - j) + (z - j_1)s' = (a + zs') - (j + j_1 s') \in U(R) \cap S = U(S)$. So $s \in csr(S)$.

Conversely, let $a = s + j \in R$ where $s \in csr(S)$ and $j \in J$ and assume that $Ra + Rb = R$. Then $xa + yb = 1$ for some $x, y \in R$. Write $b = s' + j'$, $x = s_1 + j_1$ and $y = s_2 + j_2$, where $s', s_1, s_2 \in S$ and $j', j_1, j_2 \in J$. Then $s_1 s + s_2 s' = 1 - (j_1 s + j_2 s' + xj + yj') \in U(R) \cap S = U(S)$, so $Ss + Ss' = S$. Thus, $s + zs' \in U(S)$ for some $z \in Z(S)$, and so $a + zb = (s + zs') + (j + zj) \in U(R)$. Hence $a \in csr(R)$. □

The previous Proposition has some important consequences.

Corollary 4.4 *Let R, A, B be rings, M an (R, R)-bimodule, V an (A, B)-bimodule and $n \geq 1$. Then*

(1) For $\alpha := \sum_{i \geq 0} a_i t^i \in R[[t]]$, $\alpha \in csr(R[[t]])$ iff $a_0 \in csr(R)$.
(2) For $\alpha := (a, x) \in R \propto M$, $\alpha \in csr(R \propto M)$ iff $a \in csr(R)$.
(3) If $R = \begin{bmatrix} A & V \\ 0 & B \end{bmatrix}$ is the formal triangular matrix ring and $\alpha = \begin{bmatrix} a & x \\ 0 & b \end{bmatrix} \in R$, then $\alpha \in csr(R)$ iff $a \in csr(A)$ and $b \in csr(B)$.

(4) For $\alpha = (a_{ij}) \in \mathbb{T}_n(R)$, $\alpha \in csr(\mathbb{T}_n(R))$ iff $a_{ii} \in csr(R)$ for $i = 1, ..., n$.

Finally we can prove (in a similar way to the sr1 case, see [16]) that the *central sr1 condition passes to corners*. Moreover, we prove this elementwise.

Proposition 4.5 *Let $e^2 = e \in R$ and let $a \in eRe$ If $csr_R(a) = 1$ then $csr_{eRe}(a) = 1$.*

Proof Let a and b be in $R' := eRe$ and $R'a + R'b = R'$. Consider $a + 1 - e$ and b in R. We have $R'(1 - e) = 0$, so $R(a + 1 - e) + Rb \supseteq R'a + R'b \ni e$. On the other hand, $(1 - e)a = 0 = (1 - e)b$ and so $R(a + 1 - e) + Rb \ni (1 - e)(a + 1 - e) + (1 - e)b = 1 - e$. Thus, $R(a + 1 - e) + Rb \ni e + 1 - e = 1$.

Since $csr_R(a) = 1$, there is $z \in Z(R)$ such that $a + zb + 1 - e \in U(R)$. We have $(1 - (1 - e)zb)(1 + (1 - e)zb) = 1 = (1 + (1 - e)zb)(1 - (1 - e)zb)$, so $1 - (1 - e)zb$ is a unit of R, whence $(a + zb + 1 - e)(1 - (1 - e)zb) = a + ezb + 1 - e \in U(R)$. Therefore (indeed, $U(eRe) = (eRe) \cap (1 - e + U(R))$) $a + ezeb \in U(R')$. Note that since $z \in Z(R)$, clearly $eze \in Z(eRe)$. □

In closing, motivated by the well-known fact that a regular ring has stable range one (sr1) if and only if it is unit-regular, we propose the following open question for further investigation:

How can we characterize regular rings with central stable range one (csr1)?

As noted in Section 3, strongly regular rings (equivalently, Abelian regular rings) possess central stable range one. This naturally leads to the question:

Does the converse hold?

That is, are regular rings with central stable range one necessarily strongly regular?

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Declarations

Conflict of interest There are no conflict of interest in this paper.

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