RINGS OF CHARACTERISTICS 2 MAY NOT BE BOOLEAN

A ring that consists only of idempotents is called *Boolean*. That is, a ring R is Boolean iff $a^2 = a$ for every $a \in R$.

It is well-known that *every Boolean ring has characteristics 2 and is commutative.*

For any ring (possible without identity) R, the standard proof for $\text{char}(R) = 2$ is the following.

$a + a = (a + a)^2 = a^2 + a^2 + a^2 + a^2 = a + a + a + a$ implies $a + a = 0$.

However there is a shorter proof (less letters and symbols).

$a = a^2 = (-a)^2 = -a$ implies $a + a = 0$.

The converse, "every ring of characteristics 2 is Boolean", fails, that is, not all rings of characteristic 2 are Boolean.

For example, the ring $\mathbb{M}_2(\mathbb{Z}_2)$ of square 2×2 matrices, with coefficients in $\mathbb{Z}_2$, is not Boolean.

Actually, out of 16 elements, $\mathbb{M}_2(\mathbb{Z}_2)$ contains precisely 8 idempotents, 6 units and 4 nilpotents (0_2 and I_2 are counted twice).

The polynomial ring $\mathbb{Z}_2[X]$ is another example (for instance X is not idempotent).

By $U(R)$ we denote the set of units (i.e., invertible elements) of R. Maybe less know is the following fact:

Exercise 1. *If R is Boolean then $U(R) = \{1\}$.*

that is, Boolean rings have only one unit.

Proof. Suppose $u \in U(R)$. Then, multiplying the expression $u^2 = u$ by u^{-1}, we obtain $u = 1$. Thus $U(R)$ contains the unique element 1. □

Related to this subject we mention [1], where a partial converse is proved

Theorem 2. *A* finite *ring R in which $U(R)$ is as small as possible, i.e., $U(R) = \{1\}$, is a Boolean ring.*

References

[1] Rodney Coleman *Some Properties of Finite Rings*. https://arxiv.org/abs/1302.3192