A unit product

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Exercise **1.4** in [1] asks whether it is true or false "If ab is a unit, then a, b are units ?"

The statement is false in general. If the ring is not Dedekind finite, and $ab = 1 \neq ba$, then neither a nor b are units.

Conversely, if the ring is Dedekind finite, then the statement becomes true.

Indeed, assume $(ab)c = 1 = c(ab)$. Then $a(bc) = 1$ and by DF, $(bc)a = 1$. Hence a is a unit. Analogous for b (or directly from ab and a units).

Other easy cases when the statement is true.

1) $ab = ba$.

Assume $(ab)c = 1 = c(ab)$. Then $a(bc) = 1 = c(ba) = (cb)a$, so a has a left inverse and a right inverse. Hence a is a unit.

2) Both ab and ba are units.

Assume $(ab)c = 1 = c(ab)$ and $(ba)d = 1 = d(ba)$. Then $a(bc) = 1 = (db)a$, so a has a left inverse and a right inverse. Hence a is a unit.

References

[1] T. Y. Lam *Exercises in Classical Ring Theory*. Problem Books in Mathematics, Springer Verlag 1995.