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THE X-SEMIPRIMENESS OF RINGS

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Abstract. For a nonempty subset X of a ring R, the ring R is called X-semiprime if, given $a \in R$, $aXa = 0$ implies $a = 0$. This provides a proper class of semiprime rings. First, we clarify the relationship between idempotent semiprime and unit-semiprime rings. Secondly, given a Lie ideal L of a ring R, we offer a criterion for R to be L-semiprime. For a prime ring R, we characterizes Lie ideals L of R such that R is L-semiprime. Moreover, X-semiprimeness of matrix rings, prime rings (with a nontrivial idempotent), semiprime rings, regular rings, and subdirect products are studied.

1. Introduction

Throughout the paper, rings are always associative with identity. For a ring R, $U(R)$ (resp. $Id(R)$) denotes the set of all units (resp. idempotents) of R, and $Z(R)$ stands for the center of R.

A ring R is called *semiprime* if, given $a \in R$, $aRa = 0$ implies $a = 0$. Also, it is called *prime* if, given $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$. The purpose of the paper is to study a more general notion concerning semiprimeness and primeness of rings.

Definition. Let X be a nonempty subset of a ring R. The ring R is called *X-semiprime* (resp. *X-prime*) if, given $a \in R$, $aXa = 0$ implies $a = 0$ (resp. if, given $a, b \in R$, $aXb = 0$ implies either $a = 0$ or $b = 0$).

An X-semiprime (resp. X-prime) ring with $X = \{1\}$ is reduced (resp. a domain). Also, a reduced ring (resp. domain) is X-semiprime (resp. X-prime) if $1 \in X$.

Proposition 1.1. *Given a subset X of a ring R, if R is a prime X-semiprime ring, then it is X-prime.*

Proof. Indeed, let $aXb = 0$, where $a, b \in R$. Then $bxaXbxa = 0$ for all $x \in R$. The X-semiprimeness of R implies that $bxa = 0$ for all $x \in R$. By the primeness of R, either $a = 0$ or $b = 0$, as desired. □

According to Proposition 1.1, it suffices to study the X-semiprime case. In this way, we can make our statements more concise.

In [6] the case $X = U(R)$ was considered, the ring R is called *unit-semiprime* if it is $U(R)$-semiprime. This turned out to be an interesting class of semiprime rings. In this paper we first consider the special case $X = Id(R)$. A ring R is called *idempotent semiprime* if it is $Id(R)$-semiprime. Let X^+ denote the additive

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subgroup of R generated by X. Clearly, a ring R is X-semiprime if and only if it is X^+-semiprime. Let $E(R) := Id(R)^+$. Thus the $E(R)$-semiprimeness of R just means that R is idempotent semiprime.

In this paper we will be concerned with the following

Problem 1.2. *Given a semiprime or prime ring R, find subsets X of R such that R is X-semiprime, that is, given $a \in R$,*

$$aXa = 0 \implies a = 0.$$

We organize the paper as follows.

In §2 it is proved that every idempotent semiprime ring is unit-semiprime (see Theorem 2.1). The converse is in general not true by an example from A. Smoktunowicz in [21]. Applying Theorem 2.1 we give an example of a prime ring, which is not idempotent semiprime (see Theorem 2.3).

In §3 we establish a criterion for Lie ideals L of a semiprime (resp. prime) ring R such that R is L-semiprime (see Theorem 3.6) and, in particular, matrix rings are studied. We also conclude that idempotent semiprime is not a Morita invariant property.

In §4 we give a complete characterization of the L-semiprimeness of a prime ring R for any given Lie ideal L (see Theorem 4.5). Also, in a prime ring R possessing a nontrivial idempotent, its additive subgroups X, which are invariant under all special automorphisms of R, are characterized by Chuang's theorem and hence R is X-prime [3] (see Theorem 4.7).

In §5 we characterize the $d(R)$-semiprimeness of a given prime ring R, where d is a derivation of R. Moreover, we also study the $d(L)$-semiprimeness of R for L a Lie ideal of R. See Theorems 5.2 and 5.7.

In §6 and §7 we are concerned with the problem whether, given a Lie ideal N of a semiprime ring R, $\ell_R(N) = 0$ is a sufficient condition for R to be N-semiprime, where $\ell_R(N)$ denotes the left annihilator of N in R. It is in general not true. However, it is indeed true if either $N = [E(R), R]$ or $N = [L, R]$ when R is 2-torsion free and L is a Lie ideal of R (see Theorems 6.1 and 6.4).

In §8 it is proved that every regular ring is idempotent semiprime but is in general not $[E(R), R]$-semiprime (see Theorem 8.2 and Example 8.3). Finally, in §9, we study the subdirect product properties of X-semiprime rings.

Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text.

2. Idempotent semiprime

We begin with clarifying the relationship between "idempotent semiprime" and "unit-semiprime".

Theorem 2.1. *If R is an $E(R)$-semiprime ring, then it is $U(R)$-semiprime.*

Proof. Let $a \in R$ be such that $aua = 0$ for all $u \in U(R)$. In particular, $a^2 = 0$. Let $e = e^2 \in R$ and $x \in R$. Since $1 + ex(1-e), 1 + (1-e)xe \in U(R)$, we have

$$a\big(1 + ex(1-e)\big)a = 0 = a\big(1 + (1-e)xe\big)a.$$

Thus $aeR(1-e)a = 0$ and $a(1-e)Rea = 0$. The semiprimeness of R implies that $(1-e)ae = 0 = ea(1-e)$ and so $ae = ea$. Hence $aea = ea^2 = 0$, that is, $aE(R)a = 0$ and so $a = 0$, as desired. □

The converse implication of Theorem 2.1 does not hold.

Example 2.2. Let K be a countable field and let S be the simple nil K-algebra constructed by A. Smoktunowicz in [21]. Let R be the unital algebra obtained by adjoining unity to S using K. Then R is a local prime ring with maximal ideal S and every element of R is a sum of (two) units. Thus R is unit-prime. However, R is not idempotent prime as it has only trivial idempotents and it is not a domain.

The following offers a prime ring which is not idempotent semiprime.

Theorem 2.3. *Let $R = k\langle x, y | x^2 = 0\rangle$, where k is a field. Then R is a prime ring which is not idempotent semiprime.*

Proof. It is known that R is a prime ring (see [7, p.92]). In view of [6, p.28], R is not unit-semiprime. It follows from Theorem 2.1 that R is not idempotent semiprime. □

In fact, $Id(R) = \{0, 1\}$ for the ring R in Theorem 2.3. It is an immediate consequence of Theorem 3.13.

3. Lie ideals

Let R be a ring. Given $x, y \in R$, let $[x, y] := xy - yx$ denote the additive commutator of x and y. Also, let A, B be subsets of R. We denote $[A, B]$ (resp. AB) the additive subgroup of R generated by all $[a, b]$ (resp. ab) for $a \in A$ and $b \in B$. Also, let $\mathcal{I}(A)$ be the ideal of R generated by A.

An additive subgroup L of R is called a *Lie ideal* of R if $[L, R] \subseteq L$. Given Lie ideals L, K of R, it is known that both $[L, K]$ and KL are Lie ideals of R. Recall that, given a subset A of R, we denote by A^+ the additive subgroup of R generated by A.

Lemma 3.1. *Let R a ring. Then $E(R)$ is a Lie ideal of R and $[E(R), R] \subseteq U(R)^+$.*

Proof. Indeed, let $e = e^2 \in R$ and $x \in R$. Then $e + ex(1-e)$ and $e + (1-e)xe$ are idempotents of R. So

$$[e, x] = e + ex(1-e) - (e + (1-e)xe) \in E(R).$$

Hence $[E(R), R] \subseteq E(R)$. Thus $E(R)$ is a Lie ideal of R.

On the other hand, $1 + ex(1-e)$ and $1 + (1-e)xe$ are units of R. Thus

$$[e, x] = 1 + ex(1-e) - (1 + (1-e)xe) \in U(R)^+.$$

Hence $[E(R), R] \subseteq U(R)^+$. □

Clearly, if $X \subseteq Y \subseteq R$, then the X-semiprimeness of R implies that R is Y-semiprime. By Lemma 3.1 and Theorem 2.1, we have the following

Proposition 3.2. *If a ring R is $[E(R), R]$-semiprime, then R is $E(R)$-semiprime and hence is $U(R)$-semiprime.*

A natural question is the following

Problem 3.3. *Let R be a semiprime ring. Find a sufficient and necessary condition for R to be $[E(R), R]$-semiprime.*

We note that Problem 3.3 is completely answered by Theorem 7.11 below. Motivated by Proposition 3.2, we proceed to study the L-semiprimeness of a ring R, where L is a Lie ideal of R. The first step is to establish a criterion for a ring R to be L-semiprime (see Theorem 3.6).

Lemma 3.4. *If L is a Lie ideal of a ring R, then $\mathcal{I}([L,L]) \subseteq L + L^2$.*

See, for instance, [16, Lemma 2.1]) for its proof. Given a subset A of a ring R, let $\widetilde{A}$ denote the subring of R generated by A. We continue with the following

Lemma 3.5. *Let R be a semiprime ring with a Lie ideal L. Suppose that $aLb = 0$, where $a, b \in R$. The following hold:*
(i) $a\mathcal{I}([L,L])b = 0$ and $a\widetilde{L}b = 0$;
(ii) If R be a prime ring and $[L,L] \neq 0$, then either $a = 0$ or $b = 0$.

Proof. (i) Since $a[L, RaL]b = 0$ and $aLb = 0$, we get $aRaL^2b = 0$. The semiprimeness of R implies $aL^2b = 0$ and so $a(L + L^2)b = 0$. By Lemma 3.4, we get $a\mathcal{I}([L,L])b = 0$. Note that $L + L^2$ is also a Lie ideal of R. Repeating the same argument, we finally get $a\widetilde{L}b = 0$ as $\widetilde{L} = \sum_{i=1}^{\infty} L^i$.
(ii) It follows directly from (i) and the primeness of R. □

The following gives a criterion for a ring R to be L-semiprime.

Theorem 3.6. *Let R be a ring, and let L be a Lie ideal of R. Then:*
(i) If R is semiprime and $\widetilde{L} = R$, then R is L-semiprime;
(ii) If R is prime and $[L,L] \neq 0$, then R is L-prime.

Proof. (i) Let $aLa = 0$, where $a \in R$. By Lemma 3.5 (i), $a\widetilde{L}a = 0$ and so $aRa = 0$. The semiprimeness of R implies $a = 0$.
(ii) Let $aLb = 0$, where $a, b \in R$. By Lemma 3.5 (ii), either $a = 0$ or $b = 0$, as desired. □

The following lemma is well-known. For the convenience of the reader, we give its proof.

Lemma 3.7. *Let R be a noncommutative prime ring, and let I, J be nonzero ideals of R. Then $\big[[I,J],[I,J]\big] \neq 0$.*

Proof. Replacing I, J by $I \cap J$, we may assume that $I = J$. By [8, Lemma 1.5] for the prime case, if $\big[a,[I,I]\big] = 0$ where $a \in R$ then $a \in Z(R)$.
Suppose that $\big[[I,I],[I,I]\big] = 0$. We have $[I,I] \subseteq Z(R)$ and so $\big[[I,I],R\big] = 0$. Thus $R \subseteq Z(R)$. That is, R is commutative, a contradiction. □

The following is a consequence of Theorem 3.6 (ii) and Lemma 3.7.

Theorem 3.8. *Let R be a noncommutative prime ring. If I and J are nonzero ideals of R, then R is $[I,J]$-prime.*

Lemma 3.9. *Let R be a ring. We have*
(i) If A is an additive subgroup of R, then $[A,R] = [\widetilde{A},R]$;
(ii) If L is a Lie ideal of R, then $[\mathcal{I}([L,L]),R] \subseteq [L,R] \subseteq L$.

Proof. (i) Let $a_1, \ldots, a_n \in A$ and $x \in R$. By induction on n, we get

$$[a_1a_2\cdots a_n, x] = [a_2\cdots a_n, xa_1] + [a_1, a_2\cdots a_nx] \in [A,R].$$

Thus $[A,R] = [\widetilde{A},R]$.

(ii) By Lemma 3.4 and (i), we have

$$[\mathcal{I}([L,L]),R] \subseteq [L+L^2,R] \subseteq [\widetilde{L},R] = [L,R] \subseteq L,$$

as desired. □

The following is a slight generalization of [16, Theorem 1.4] with $V = R$.

Theorem 3.10. *Let R be a ring with a Lie ideal L. If $\mathcal{I}([L,L]) = R$, then $[L,R] = [R,R]$ and $R = [R,R]^2$. In addition, if R is semiprime, then it is $[L,R]$-semiprime.*

Proof. By Lemmas 3.4 and 3.9 (i), we have

$$[R,R] = [\mathcal{I}([L,L]),R] \subseteq [L+L^2,R] = [L,R].$$

Hence $[R,R] = [L,R]$. Since $\mathcal{I}([L,L]) = R$, it is clear that $\mathcal{I}([R,R]) = R$ and so, by [16, Theorem 1.4], $R = [R,R] + [R,R]^2$. Thus we have

$$\begin{aligned} [R,R] &= \big[[R,R]+[R,R]^2, [R,R]+[R,R]^2\big] \\ &= \big[[R,R],[R,R]\big] + \big[[R,R]+[R,R]^2, [R,R]^2\big] \\ &= \big[[R,R],[R,R]\big] + \big[R,[R,R]^2\big] \\ &\subseteq [R,R]^2 + [R,R]^2 \\ &= [R,R]^2, \end{aligned}$$

where we have used the fact that $[R,R]^2$ is a Lie ideal of R. Hence

$$R = [R,R] + [R,R]^2 \subseteq [R,R]^2 + [R,R]^2 = [R,R]^2,$$

as desired.

In addition, assume that R is semiprime. Since

$$R = [R,R]^2 = [L,R]^2 \subseteq \widetilde{[L,R]},$$

we have $R = \widetilde{[L,R]}$. In view of Theorem 3.6(i), R is $[L,R]$-semiprime. □

Remark 3.11. (i) In Theorem 3.10, R is in general not $[L,L]$-semiprime. For instance, let $R := \mathbb{M}_2(F)$, where F is a field of characteristic 2, and let $L := [R,R]$. In view of [16, Lemma 2.3], we have $0 \neq [L,L] \subseteq F$. Thus $\mathcal{I}([L,L]) = R$ but R is not $[L,L]$-semiprime as R is not a domain.

(ii) In Theorem 3.10, if the assumption $\mathcal{I}([L,L]) = R$ is replaced by $\mathcal{I}([L,R]) = R$, we cannot conclude that R is $[L,R]$-semiprime. See Remark 6.2 (i) and (ii) in the next section, and a related result [17, Theorem 1.1].

The equality $[L,L] = [L,R]$ is in general not true. However, we always have the following

Lemma 3.12. *Let R be a ring. Then $[E(R),R] = [E(R),E(R)]$.*

Proof. Indeed, let $e = e^2 \in R$ and $x \in R$. Then

$$[e,x] = \big[e,[e,[e,x]]\big] \in \big[E(R),[E(R),R]\big] \subseteq [E(R),E(R)].$$

So $[E(R),E(R)] = [E(R),R]$. □

Theorem 3.13. *Let R be a prime ring with a nontrivial idempotent. Then R is $[E(R),R]$-prime.*

Proof. Let $E := E(R)$. Clearly, we have $E \nsubseteq Z(R)$. By Lemma 3.12, we have $[E, E] = [E, R] \neq 0$. Let $I := \mathcal{I}([E, E])$, a nonzero ideal of R. It follows from Lemma 3.9 (ii) that

$$0 \neq [I, R] = [\mathcal{I}([E, E]), R] \subseteq [E, R] = [E, E].$$

By Lemma 3.7, $\big[[E, R], [E, R]\big] \neq 0$. It follows from Theorem 3.6 (ii) that R is $[E, R]$-prime. □

Theorem 3.13 is also an immediate consequence of Theorem 7.11.

Corollary 3.14. *Every prime $E(R)$-semiprime ring R is either a domain or an $[E(R), R]$-prime ring.*

Proof. Assume that R is not a domain. Since R is a prime ring, it contains nonzero square zero elements. Let $a^2 = 0$, where $0 \neq a \in R$. Since R is $E(R)$-semiprime, this implies that $E(R) \neq \{0, 1\}$. It follows from Theorem 3.13 that R is $[E(R), R]$-prime. □

We now apply Theorem 3.10 to the case of matrix rings.

Theorem 3.15. *Let $R := \mathbb{M}_n(A)$, where A is a semiprime ring and $n > 1$. Then R is $[E(R), R]$-semiprime.*

Proof. Let $E := E(R)$. By [14, Theorem 2.1], we have $\mathcal{I}([E, E]) = R$. Since A is a semiprime ring, so is R. By Theorem 3.10, R is $[E, R]$-semiprime. □

The following corollary is a consequence of Theorem 3.15 and Proposition 3.2.

Corollary 3.16. *Let $R := \mathbb{M}_n(A)$, where A be a semiprime ring and $n > 1$. Then R is idempotent semiprime.*

The above corollary is also a generalization of [6, Theorem 10] which allows to prove easily the following result.

Corollary 3.17. *The idempotent semiprime property does not pass to corners.*

Proof. Choose a semiprime ring R, which is not idempotent semiprime, and $n > 1$ an integer. Then $e_{11}\mathbb{M}_n(R)e_{11} \cong R$ and so $\mathbb{M}_n(R)$ is idempotent semiprime but R itself is not. □

Corollary 3.18. *Idempotent semiprime is not a Morita invariant property of rings.*

Throughout, we use the following notation. Let R be a semiprime ring. We can define its *Martindale symmetric ring of quotients* $Q(R)$. The center of this ring, denoted by C, is called the *extended centroid* of R. It is known that $Q(R)$ itself is a semiprime ring and C is a regular self-injective ring. Moreover, C is a field iff R is a prime ring. We refer the reader to the book [1] for details.

4. Prime rings

In this section, given a Lie ideal L of a prime ring R, we obtain a complete characterization for R to be L-prime (see Theorem 4.5). The following is a special case of [9, Theorem], which will be used in the proofs below.

Lemma 4.1. *Let R be a prime ring, $a, b \in R$ with $b \notin Z(R)$. Suppose that $[a, [b, x]] = 0$ for all $x \in R$. Then $a^2 \in Z(R)$. In addition, if $\operatorname{char} R \neq 2$, then $a \in Z(R)$.*

The following is well-known.

Lemma 4.2. *Let R be a prime ring. If $[a, R] \subseteq Z(R)$ where $a \in R$, then $a \in Z(R)$.*

Lemma 4.3. *Let R be a prime ring. If $a \in RC \setminus C$, then $\dim_C [a, RC] > 1$.*

Proof. Suppose not. We have $\dim_C [a, RC] = 1$ and $[a, RC] = Cw$ for some $w \in [a, RC]$. In particular, $\big[w, [a, RC]\big] = 0$. In view of Lemma 4.1, we have $w^2 \in C$. Thus $w[a, RC] = Cw^2 \subseteq C$.

Let $x \in RC$. Then $w[a, xa] = w[a, x]a \in C$. Since $w[a, x] \in C$, we get $w[a, x] = 0$. Thus $w[a, R] = 0$. The primeness of R forces $w = 0$ and so $a \in C$. This is a contradiction. □

Definition. A noncommutative prime ring R is called *exceptional* if both $\operatorname{char} R = 2$ and $\dim_C RC = 4$. Otherwise, R is called *non-exceptional*.

A Lie ideal L of a ring R is called *proper* if $[I, R] \subseteq L$ for some nonzero ideal I of R. We need the following lemma (see [11, Lemma 7]).

Lemma 4.4. *Let R be a prime ring with a Lie ideal L. Then L is noncentral iff $[L, L] \neq 0$ unless R is exceptional.*

Clearly, if L is a nonzero central Lie ideal of R, then R is L-prime if and only if R is a domain.

Theorem 4.5. *Let R be a prime ring, and let L be a noncentral Lie ideal of R. If R is not a domain, then R is L-prime if and only if one of the following holds:*

(i) L is a proper Lie ideal of R;

(ii) R is exceptional, $[L, L] = 0$, $\dim_C LC = 2$ and $LC = [a, RC]$, where $a \in L$ such that $a + \beta$ is invertible in RC for all $\beta \in C$.

Proof. Clearly, R is not commutative. By Lemma 3.9 (ii), $[\mathcal{I}([L, L]), R] \subseteq L$.

"$\Longrightarrow$": Suppose that R is L-prime. If $\mathcal{I}([L, L]) \neq 0$, then L is a proper Lie ideal of R and (i) holds. Assume next that $\mathcal{I}([L, L]) = 0$, that is, $[L, L] = 0$. Since L is a noncentral Lie ideal of R, it follows from Lemma 4.4 that R is exceptional. Also, R is not a domain and so $RC \cong \mathbb{M}_2(C)$. Clearly, LC is a commutative Lie ideal of RC.

Since LC is noncentral, $0 \neq [LC, RC] \subseteq LC$ and, by Lemma 4.2, $[LC, RC] \nsubseteq C$. Choose a nonzero element $a \in [LC, RC] \setminus C$. Then $0 \neq [a, RC] \subseteq LC$. It is well-known that $Z(R) \neq 0$ and C is the quotient field of $Z(R)$. Hence we may choose $a \in L$. It follows from Lemma 4.3 that $\dim_C [a, RC] > 1$.

Suppose that $\dim_C LC = 3$. Then $RC = LC + Cz$ for some $z \in RC$. Thus

$$[RC, RC] = [LC + Cz, LC + Cz] = [LC, Cz] \subseteq LC.$$

In particular, $\big[[RC, RC], [RC, RC]\big] = 0$, a contradiction (see Lemma 3.7). Hence $LC = [a, RC]$ and $\dim_C LC = 2$.

Suppose on the contrary that $a + \beta$ is not invertible in RC for some $\beta \in C$. Then we can choose nonzero $b, c \in RC$ such that $b(a + \beta) = 0$ and $(a + \beta)c = 0$. We may choose $b, c \in R$. Given $x \in RC$, we have

$$b[a, x]c = b[a + \beta, x]c = b(a + \beta)xc - bx(a + \beta)c = 0.$$

Hence $b[a, RC]c = 0$. That is, $bLCc = 0$ and so $bLc = 0$. This is a contradiction as R is L-prime. So (ii) holds.

"$\Longleftarrow$": Suppose that (i) holds. Then $[I, R] \subseteq L$ for some nonzero ideal I of R. In view of Theorem 3.8, R is $[I, R]$-prime and so it is L-prime. We next consider the case (ii). Let $b, c \in R$ be such that $bLc = 0$. Then $bLCc = 0$ and so $b[a, x]c = 0$ for all $x \in RC$. That is,

$$baxc - bxac = 0$$

for all $x \in RC$. Suppose first that ba and b are C-independent. In view of [19, Theorem 2], $c = 0$ follows, as desired. Suppose next that ba and b are C-dependent. That is, there exists $\beta \in C$ such that $ba = \beta b$. So $b(a-\beta) = 0$. Since, by assumption, $a - \beta$ is invertible in RC and we get $b = 0$. Hence R is L-prime. □

Remark 4.6. The case (ii) of Theorem 4.5 indeed occurs. Let $R := \mathbb{M}_2(F)$, where F is a field of characteristic 2.

(i) Let

$$L = \{ \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \mid \alpha, \beta \in F\} = [a, R],$$

where $a := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then L is a noncentral Lie ideal of R satisfying $[L, L] = 0$. Since $a \in L$ and $a^2 = 0$, we get $aLa = 0$ but $a \neq 0$. Thus, R is not L-prime.

(ii) We choose F such that there exists $\eta \in F$ satisfying $\eta \notin F^{(2)} := \{\mu^2 \mid \mu \in F\}$. Let

$$L = \{ \begin{bmatrix} \alpha & \beta \\ \beta\eta & \alpha \end{bmatrix} \mid \alpha, \beta \in F\} = [a, R],$$

where $a := \begin{bmatrix} 1 & 1 \\ \eta & 1 \end{bmatrix}$. Then L is a noncentral Lie ideal of R satisfying $[L, L] = 0$. Note that $a + \beta$ is invertible for all $\beta \in F$. Hence R is L-prime.

We follow the notation given in [3]. A subset X of a ring R is said to be invariant under *special* automorphisms if $(1+t)X(1+t)^{-1} \subseteq X$ for all $t \in R$ such that $t^2 = 0$. Clearly, if $X \subseteq R$ is invariant under special automorphisms, then so is X^+. Also, R is X-prime if and only if it is X^+-prime.

Theorem 4.7. *Let R be a prime ring with a nontrivial idempotent and let X be a subset of R invariant under special automorphisms. If $X \nsubseteq Z(R)$, then R is X-prime.*

Proof. Without loss of generality we can replace X by X^+ and assume that X is an additive subgroup of R. Clearly, R is not commutative.

Case 1: R is non-exceptional. Then, as $X \nsubseteq Z(R)$, we can apply [3, Theorem 1] to get that X contains a proper Lie ideal L of R. In view of Theorem 4.5 (i), R is L-prime and hence is X-prime.

Case 2: R is exceptional. It follows from [3, Lemma 11] that $XZ(R)$ contains a proper Lie ideal of R. Thus, using Theorem 4.5 (i) again, we obtain that R is $XZ(R)$-prime and hence is X-prime. □

The following are natural examples of X: potent elements, potent elements of a fixed degree, nilpotent elements, nilpotent elements of a fixed degree (in particular, elements of square zero), $[E(R), R]$, $U(R)$, $E(R)$ etc. Moreover, if A, B are invariant under special automorphisms, then so are AB and $[A, B]$.

Let $N(R)$ denote the set of all nilpotent elements of R. We end this section with the following corollary, which is a consequence of Theorem 4.7.

Corollary 4.8. *Let R be a ring possessing a nontrivial idempotents. The following are equivalent:*

(i) R is a prime ring;
(ii) R is $U(R)$-prime;
(iii) R is $N(R)$-prime.

5. Derivations

By a *derivation* of a ring R we mean an additive map $d\colon R \to R$ satisfying

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in R$. A derivation d of R is called *inner* if there exists $b \in R$ such that $d(x) = [b, x]$ for all $x \in R$. In this case, we denote $d = \mathrm{ad}_b$. Otherwise, d is called *outer*. In this section we characterize the $d(R)$-semiprimeness of a given prime ring R. Moreover, we also study the $d(L)$-semiprimeness of R for L a Lie ideal of R.

Let R be a prime ring with a derivation d. It is known that d can be uniquely extended to a derivation, denoted by d also, of $Q(R)$ (by applying a standard argument). We say that d is $\mathfrak{X}$-*inner* if d is inner on $Q(R)$ and d is called $\mathfrak{X}$-*outer*, otherwise.

We need a preliminary proposition due to Kharchenko (see [10, Lemma 2]).

Proposition 5.1. *Let R be a prime ring with a derivation δ. Suppose that there exist finitely many $a_i, b_i, c_j, d_j \in Q(R)$ such that*

$$\sum_i a_i\delta(x)b_i + \sum_j c_jxd_j = 0$$

for all $x \in R$. If δ is $\mathfrak{X}$-outer, then $\sum_i a_ixb_i = 0 = \sum_j c_jxd_j$ for all $x \in R$.

Given $b \in Q(R)$, let $\ell_R(b) := \{x \in R \mid xb = 0\}$, the left annihilator of b in R. Similarly, we denote by $r_R(b)$ the right annihilator of b in R.

We are now ready to prove the first main theorem in this section.

Theorem 5.2. *Let R be a noncommutative prime ring with a derivation d. Then R is $d(R)$-semiprime iff one of the following conditions holds*

(i) d is $\mathfrak{X}$-outer;
(ii) $d = \mathrm{ad}_b$ for some $b \in Q(R)$, and for any $\beta \in C$, either $\ell_R(b + \beta) = 0$ or $r_R(b + \beta) = 0$.

Proof. "$\Longrightarrow$": Suppose that R is $d(R)$-semiprimeness. Assume that d is $\mathfrak{X}$-inner. Thus there exists $b \in Q(R)$ such that $d(x) = [b, x]$ for all $x \in R$. We claim that given any $\beta \in C$, either $\ell_R(b + \beta) = 0$ or $r_R(b + \beta) = 0$. Otherwise, there exist $\beta \in C$ and nonzero elements $a, c \in R$ such that $a(b + \beta) = 0 = (b + \beta)c$. By the primeness of R, $w := cya \neq 0$ for some $y \in R$. Then

$$wd(z)w = w[b + \beta, z]w = cya(b + \beta)zw - wz(b + \beta)cya = 0$$

for all $z \in R$. That is, $wd(R)w = 0$ with $w \neq 0$. So R is not $d(R)$-semiprimeness, a contradiction.

"$\Longleftarrow$": (i) Assume that d is $\mathfrak{X}$-outer. Let $ad(x)a = 0$ for all $x \in R$. By Proposition 5.1, we get $aya = 0$ for all $y \in R$. The primeness of R implies $a = 0$. This proves that R is $d(R)$-semiprime.

(ii) Let $ad(x)a = 0$ for all $x \in R$. Since $d(x) = [b, x]$ for all $x \in R$, we get

$$0 = ad(x)a = abxa - axba$$

for all $x \in R$. In view of [19, Theorem 2], there exists $\beta \in C$ such that $ab = -\beta a$, i.e. $a \in \ell_R(b+\beta)$. Thus $ax(b+\beta)a = 0$ for all $x \in R$. Hence $(b+\beta)a = 0$, i.e., $a \in r_R(b+\beta)$. Since either $\ell_R(b+\beta) = 0$ or $r_R(b+\beta) = 0$, we get $a = 0$. Hence R is $d(R)$-semiprime. □

As a direct application of the above theorem we get the following

Corollary 5.3. Let $R := \mathbb{M}_m(D)$, where D is a noncommutative division ring, $m \geq 1$, and let $d = \mathrm{ad}_b$, where $b := \sum_{i=1}^m \mu_i e_{ii}$, where $\mu_i \in D$ for all i. Then R is $d(R)$-semiprime iff $\mu_i \notin Z(D)$ for any i.

Let $x \in \mathbb{M}_m(F)$, where F is a field. We denote by $\det(x)$ the determinant of x.

Corollary 5.4. *Let $R := \mathbb{M}_m(F)$, where F is a field, $m > 1$, and let d be a derivation of R. Then R is $d(R)$-semiprime iff either d is outer or $d = \mathrm{ad}_b$ for some $b \in R$ such that* $\det(b+\beta) \neq 0$ *for any $\beta \in F$.*

In addition, if F is algebraically closed, then R is $d(R)$-semiprime iff d is outer.

In Corollary 5.4, let $\overline{F}$ be the algebraic closure of the field F. It is known that the matrix b can be upper triangularizable in $\mathbb{M}_m(\overline{F})$, that is, there exists a unit u of $\mathbb{M}_m(\overline{F})$ such that $ubu^{-1} = \sum_{1 \leq i \leq j \leq n} \mu_{ij} e_{ij}$, where $\mu_{ij} \in \overline{F}$. Thus $\det(b+\beta) \neq 0$ for any $\beta \in F$ iff $\mu_{ii} \in \overline{F} \setminus F$ for all i.

Motivated by the above two results, it is natural to raise the following

Problem 5.5. *Let $R := \mathbb{M}_m(D)$, where D is a noncommutative division ring, $m \geq 1$. Characterize elements $b \in R$ such that $b + \beta \in U(R)$ for any $\beta \in Z(D)$.*

The problem seems to be related to the triangularizability of the element b. We next deal with the $d(L)$-semiprimeness of a prime ring R, where L is a Lie ideal of R and d is a derivation of R.

Lemma 5.6. *Let R be a prime ring with a nonzero ideal I, and $a_i, b_i \in Q(R)$ for $i = 1, \ldots, m$. Then:*

(i) $\sum_{i=1}^m a_i x b_i = 0$ for all $x \in I$ iff $\sum_{i=1}^m b_i x a_i = 0$ for all $x \in I$;

(ii) $\sum_{i=1}^m a_i w b_i = 0$ for all $w \in [I, I]$ iff $\sum_{i=1}^m b_i x a_i \in C$ for all $x \in R$.

Proof. (i) If follows directly from [18, Corollary 2.2].

(ii) Applying (i) we have

$$
\begin{aligned}
\sum_{i=1}^m a_i[x,y]b_i = 0 \ \forall x, y \in I \quad &\Leftrightarrow \quad \sum_{i=1}^m a_i x(yb_i) - (a_i y)xb_i = 0 \ \forall x, y \in I \\
&\Leftrightarrow \quad \sum_{i=1}^m (yb_i)xa_i - b_i x(a_i y) = 0 \ \forall x, y \in I \\
&\Leftrightarrow \quad \big[y, \sum_{i=1}^m b_i x a_i\big] = 0 \ \forall x, y \in I \\
&\Leftrightarrow \quad \sum_{i=1}^m b_i x a_i \in C \ \forall x \in I \\
&\Leftrightarrow \quad \sum_{i=1}^m b_i x a_i \in C \ \forall x \in R,
\end{aligned}
$$

where the last equivalence holds as R and I satisfy the same GPIs with coefficients in $Q(R)$ (see [4, Theorem 2]). ☐

The following is the second main result in this section.

Theorem 5.7. *Let R be a non-exceptional prime ring, not a domain, with a non-central Lie ideal L, and let d be a derivation of R. Then:*
(i) If d is $\mathfrak{X}$-outer, then R is $d(L)$-semiprime;
(ii) If d is $\mathfrak{X}$-inner, then R is $d(L)$-semiprime iff it is $d(R)$-semiprime.

Proof. Since R is non-exceptional, either $\operatorname{char} R \neq 2$ or $\dim_C RC > 4$. In view of Lemma 4.4, we have $[L, L] \neq 0$. Thus, by Theorem 3.6 (ii), R is L-semiprime.

(i) Assume that $ad(L)a = 0$, where $a \in R$. Let $x \in L$ and $r \in R$. Then $ad([x, r])a = 0$ and so

$$a\big([d(x), r] + [x, d(r)]\big)a = 0.$$

Since d is $\mathfrak{X}$-outer, applying Proposition 5.1 we get

$$a\big([d(x), r] + [x, z]\big)a = 0$$

for all $x \in L$ and all $r, z \in R$. In particular, $a[x, z]a = 0$ for all $x \in L$ and $z \in R$. That is, $a[L, R]a = 0$.

In particular, $a[L, RaR]a = 0$. Since $a[L, R](aR)a = 0$, we get $aR[L, aR]a = 0$. By the primeness of R, we have $[L, aR]a = 0$ and so $[L, a]Ra = 0$. Thus $[a, L] = 0$. By the fact that $a[L, R]a = 0$ and $[L, R] \subseteq L$, we have $a^2[L, R] = 0$. This implies $a^2 = 0$ as L is noncentral.

It follows from $a[L, aR]a = 0$ and $a^2 = 0$ that $aLaRa = 0$, implying $aLa = 0$. Since R is L-semiprime, we get $a = 0$, as desired.

(ii) Assume that d is $\mathfrak{X}$-inner. Clearly, if R is $d(L)$-semiprime, then it is $d(R)$-semiprime. Conversely, assume that R is $d(R)$-semiprime. Since d is $\mathfrak{X}$-inner, there exists $b \in Q(R)$ such that $d(x) = [b, x]$ for all $x \in R$. In view of Lemma 4.4, $[L, L] \neq 0$. Hence, by Lemma 3.9 (i), we have $0 \neq [K, R] \subseteq L$, where $K := \mathcal{I}([L, L])$.

Let $ad(L)a = 0$, where $a \in R$. The aim is to prove $a = 0$. Then $a[b, x]a = 0$ and so

$$abxa = axba \tag{1}$$

for all $x \in [K, K] \subseteq L$. Since R is non-exceptional, the proof is divided into the following two cases.

Case 1: $\dim_R C > 4$. Applying Lemma 5.6 (ii) to Eq.(1), we have

$$ayab - baya \in C \tag{2}$$

for all $y \in R$.

Suppose first that $ay_0ab - bay_0a \neq 0$ for some $y_0 \in R$. Applying [5, Fact 3.1] to Eq.(2), we get $Q(R) = RC$ and $\dim_C RC < \infty$. It follows from [5, Theorem 1.1] that $\dim_C RC \leq 4$, a contradiction. Thus $ayab - baya = 0$ for all $y \in R$. In view of Lemma 5.6 (i), $abya - ayba = 0$ for all $y \in R$. That is, $ad(R)a = 0$. Since R is $d(R)$-semiprime, we get $a = 0$, as desired.

Case 2: $\dim_R RC = 4$ and $\operatorname{char} R \neq 2$. Since R is not a domain, we have $RC \cong \mathbb{M}_2(C)$. Note that $KC = RC$ in this case. Moreover, $[KC, KC] + C = RC$ as $\operatorname{char} R \neq 2$. Clearly, Eq.(1) holds for all $x \in [KC, KC]$.

Let $y \in RC$. Then $y = x + \beta$ for some $x \in [KC, KC]$ and $\beta \in C$. By Eq.(1) we have

$$abya = abxa + ab\beta a = axba + \beta aba = a(x + \beta)ba = ayba.$$

Thus $a[b,y]a = 0$ for all $y \in RC$. In particular, $ad(R)a = 0$. Since R is $d(R)$-semiprime, we get $a = 0$, as desired. □

By Theorems 5.7 and 5.2, we have the following

Corollary 5.8. *Let R be a non-exceptional prime ring, not a domain, with a noncentral Lie ideal L, and let d be a derivation of R. Then R is $d(L)$-semiprime if one of the following conditions holds:*

(i) d is $\mathfrak{X}$-outer;

(ii) There exists $b \in Q(R)$ such that $d(x) = [b,x]$ for all $x \in R$. Moreover, given any $\beta \in C$, either $\ell_R(b+\beta) = 0$ or $r_R(b+\beta) = 0$.

Example 5.9. Let $R := \mathbb{M}_2(F)$, where F is a field of characteristic 2. We choose F such that there exists $\eta \in F$ satisfying $\eta \notin F^{(2)} := \{\mu^2 \mid \mu \in F\}$. Let

$$L = \{\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \mid \alpha, \beta \in F\} = [a, R],$$

where $a := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then L is a noncentral Lie ideal of R satisfying $[L,L] = 0$. Since $a \in L$ and $a^2 = 0$, we get $aLa = 0$ but $a \neq 0$. Thus, R is not L-prime. Let $d(x) := [b,x]$ for all $x \in R$, where $b := \begin{bmatrix} 1 & 1 \\ \eta & 1 \end{bmatrix}$. Then d is an inner derivation of R. Since $d(L) \subseteq L$ and R is not L-semiprime, it follows that R is not $d(L)$-semiprime. Notice that $b + \beta$ is a unit of R for all $\beta \in F$. In view of Theorem 5.2 (ii), R is $d(R)$-semiprime.

6. Semiprime rings I: main results

Recall that, given a semiprime ring R, we denote by $Q(R)$ the Martindale symmetric ring of quotients of R and by C the extended centroid of R. The following two sections are about proving Theorems 6.1 and 6.4.

Theorem 6.1. *Let R be a semiprime ring, and let L be a Lie ideal of R. Suppose that $\ell_R([L,R]) = 0$. Then there exists an idempotent $e \in C$ such that*

(i) $ex^2 \in C$ for all $x \in \widetilde{L}$, and

(ii) $(1-e)R$ is $(1-e)L$-semiprime.

In addition, if R is 2-torsion free, then $e = 0$ and so R is L-semiprime.

Remark 6.2. In Theorem 6.1, since $\ell_R([L,R]) = 0$, R is L-semiprime iff it is $[L,R]$-semiprime (see Corollary 7.4 below). However, we cannot conclude that R is $[L,R]$-semiprime unless R is 2-torsion free (see Remark 4.6 (i)).

Applying Corollary 7.4 and the 2-torsion free case of Theorem 6.1, we have the following

Corollary 6.3. *Let R be a 2-torsion free semiprime ring, and let L be a Lie ideal of R. Then R is $[L,R]$-semiprime iff $\ell_R([L,R]) = 0$.*

When B is a subset of $Id(R)$ such that B^+ is a Lie ideal of R, we can get better conclusions for arbitrary semiprime rings as follows.

Theorem 6.4. *Let R be a semiprime ring, and let B be a subset of $Id(R)$ such that B^+ is a Lie ideal of R. Then $\ell_R([B,R]) = 0$ iff R is $[B,R]$-semiprime.*

The meaning of the two results above deserves further understanding. Clearly, given a Lie ideal L of a ring R, the assumption $\ell_R(L) = 0$ is necessary to ensure the L-semiprimeness of R but is in general not sufficient. Thus Corollary 6.3 and Theorem 6.4 are indeed very special and have good conclusions. It is natural to raise the following

Problem 6.5. *Let R be a semiprime ring with a subset X, and let*

$$X^{(n)} := \{x^n \mid x \in X\},$$

where n is a positive integer.

(i) Characterize Lie ideals L of R such that R is L-semiprime iff $\ell_R(L) = 0$.

(ii) Find subsets X of R such that R is X-semiprime iff $\ell_R(X) = 0$.

(iii) Let L be a Lie ideal of R. If R is L-semiprime, is then it $L^{(n)}$-semiprime?

(iv) If $\ell_R([E(R), R]) = 0$, is then R an $[E(R), R]^{(n)}$-semiprime ring?

(v) Let R be 2-torsion free and let L be a Lie ideal of R. If $\ell_R(L, R]) = 0$, is then R a $[L, R]^{(n)}$-semiprime ring?

The following is clear.

Lemma 6.6. *Let R be a ring with subsets X, Y. Then:*

(i) If R is both X-semiprime and Y-semiprime, then R is XY-semiprime;

(ii) If R is X-semiprime, then R is X^n-semiprime for any positive integer n.

We give some examples of Problem 6.5 (ii).

Example 6.7. Let R be a semiprime ring, and let ρ be a right ideal of R.

(i) The ring R is ρ^n-semiprime iff $\ell_R(\rho) = 0$, where n is a positive integer. Indeed, it is clear that R is ρ-semiprime iff $\ell_R(\rho) = 0$. Thus, if $\ell_R(\rho) = 0$, then R is ρ-semiprime and hence is ρ^n-semiprime (see Lemma 6.6 (ii)). Conversely, assume that f is ρ^n-semiprime. Then $\ell_R(\rho^n) = 0$ and so $\ell_R(\rho) = 0$ as $\rho^n \subseteq \rho$.

(ii) Let X be a subset of R such that, for any $x \in \rho$, $x^{n(x)} \in X$ for some positive integer $n(x) \leq m$, a fixed positive integer. Assume that $aXa = 0$, where $a \in R$. Then $ax^{n(x)}a = 0$ for all $x \in R$. In view of [12, Theorem 2], we get $a\rho a = 0$. Thus R is X-semiprime iff $\ell_R(X) = 0$ iff $\ell_R(\rho) = 0$.

(iii) Let X be a subset of R such that, for any $x \in \rho$, $x^{n(x)} \in X$ for some positive integer $n(x)$. If R has no nonzero nil one-sided ideals, then R is X-semiprime iff $\ell_R(\rho) = 0$ (see [12, Theorem 1]).

7. Semiprime rings II: proofs

We begin with some preliminaries. Given an ideal I of $Q(R)$, it follows from the semiprimeness of $Q(R)$ that, for $a \in Q(R)$, $aI = 0$ iff $Ia = 0$. Thus $\ell_{Q(R)}(I)$, the left annihilator of I in $Q(R)$, is an ideal of $Q(R)$.

By an *annihilator ideal* of $Q(R)$, we mean an ideal N of $Q(R)$ such that $N = \ell_{Q(R)}(I)$ for some ideal I of $Q(R)$. The following is well-known (see, for instance, [15, Lemma 2.10]).

Lemma 7.1. *Let R be a semiprime ring. Then every annihilator ideal of $Q(R)$ is of the form $eQ(R)$ for some idempotent $e \in C$.*

Let R be a ring with a Lie ideal L. Given $x \in R$, $xLR \subseteq ([L, x] + Lx)R \subseteq LR$. This proves that LR is an ideal of R. This fact will be used in the proof below.

Lemma 7.2. *Let R be a semiprime ring, and let L be a Lie ideal of R. Then*

$$\ell_{Q(R)}(L) = \ell_{Q(R)}\big(Q(R)LQ(R)\big) = eQ(R)$$

for some idempotent $e \in C$.

Proof. Let $Q := Q(R)$. For $a, b \in Q$, it is easy to prove that $aRb = 0$ iff $aQb = 0$ (it follows from the definition of Q). It suffices to claim that, given $a \in Q$, $aL = 0$ iff $aQLQ = 0$. The converse implication is trivial. Suppose that $aL = 0$. Then $aLR = 0$. Note that LR is an ideal of the semiprime ring R. We get $LRa = 0$ and hence $LQa = 0$. So $QLQa = 0$ and the semiprimeness of Q forces $aQLQ = 0$. It follows from Lemma 7.1 that

$$\ell_Q(L) = \ell_Q(QLQ) = eQ$$

for some idempotent $e \in C$. □

Lemma 7.3. *Let R be a semiprime ring with a Lie ideal L. The following hold:*
(i) If $aLa = 0$ where $a \in R$, then $[a, L] = 0$;
(ii) If $\ell_R([L, R]) = 0$, then given $a \in R$, $a[L, R]a = 0$ implies $aLa = 0$.

Proof. (i) Since $aLa = 0$, we have $a[L, RaR]a = 0$. By the fact that $a[L, R](aR)a = 0$, we get $aR[L, aR]a = 0$. The semiprimeness of R implies that $[L, aR]a = 0$. Since $a[L, R]a = 0$, we have $[L, a]Ra = 0$. It follows from the semiprimeness of R again that $[a, L] = 0$, as desired.

(ii) Assume that $\ell_R([L, R]) = 0$ and $a[L, R]a = 0$. Since $[L, R]$ is a Lie ideal of R, it follows from (i) that $\big[a, [L, R]\big] = 0$ and so $a^2[L, R] = 0$. Hence $a^2 = 0$ as $\ell_R([L, R]) = 0$. Now, it follows from $a[L, aR]a = 0$ that $aLaRa = 0$ and so, by the semiprimeness of R, $aLa = 0$, as desired. □

As an immediate consequence of Lemma 7.3 (ii), we have the following

Corollary 7.4. *Let R be a ring with a Lie ideal L satisfying $\ell_R([L, R]) = 0$. Then R is L-semiprime iff it is $[L, R]$-semiprime.*

Clearly, if R is a prime ring and $a[b, R] = 0$, where $a, b \in R$, then either $a = 0$ or $b \in Z(R)$. The following is a consequence of Corollary 7.4.

Corollary 7.5. *Let R be a prime ring with a noncentral Lie ideal L. Then R is L-semiprime iff it is $[L, R]$-semiprime.*

The next aim is to study semiprime rings R with a Lie ideal L satisfying $\ell_R([L, R]) = 0$. We need a technical lemma.

Lemma 7.6. *Let R be a semiprime ring with a Lie ideal L, and n a positive integer. Assume that $x^n \in Z(R)$ for all $x \in \widetilde{L}$. Then $\mathcal{I}([L, L]) \subseteq Z(R)$. In addition, if R is 2-torsion free, then $L \subseteq Z(R)$.*

Proof. In view of Lemma 3.4, we have $\mathcal{I}([L, L]) \subseteq L + L^2 \subseteq \widetilde{L}$. Thus $[x^n, R] = 0$ for all $x \in \mathcal{I}([L, L])$. In view of [13, Lemma 2], we have $[x, R] = 0$ for all $x \in \mathcal{I}([L, L])$. That is, $\mathcal{I}([L, L]) \subseteq Z(R)$. In particular, $[L, L] \subseteq Z(R)$.

In addition, assume that R is 2-torsion free. Let P be a prime ideal of R such that $\operatorname{char} R/P \neq 2$. Working in $R := R/P$, we get $[\overline{L}, \overline{L}] \subseteq Z(\overline{R})$ and so $\overline{L} \subseteq Z(\overline{R})$ (see [2, Lemma 6]). That is, $[L, R] \subseteq P$. Since R is 2-torsion free, the intersection of prime ideals P of R with $\operatorname{char} R/P \neq 2$ is zero, it follows that $[L, R] = 0$ and so $L \subseteq Z(R)$, as desired. □

Let R be a semiprime ring, and let L be a Lie ideal of R. We denote $\ell_R(L) := \{a \in R \mid aL = 0\}$, the left annihilator of L in R. In view of Lemma 7.2,

$$\ell_R(L) = R \cap \ell_{Q(R)}(L),$$

implying that $\ell_R(L)$ is an ideal of R.

Proof of Theorem 6.1.

Let $Q := Q(R)$. In view of Lemma 7.2, there exists an idempotent $e \in C$ such that

$$\ell_Q\Big(\sum_{x\in\widetilde{L}} R[x^2, R]R\Big) = eQ.$$

Clearly, $ex^2 \in C$ for all $x \in \widetilde{L}$. This proves (i). We next prove (ii), i.e., $(1-e)R$ is $(1-e)L$-semiprime.

Since $\ell_R([L, R]) = 0$, it is clear that

$$\ell_{(1-e)R}([(1-e)L, (1-e)R]) = 0. \tag{3}$$

Let $b(1-e)Lb = 0$, where $b = (1-e)a \in (1-e)R$ for some $a \in R$. The aim is to prove $b = 0$. By $b \in (1-e)R$, we get $a(1-e)La = 0$. Note that $(1-e)L$ is a Lie ideal of the semiprime ring $(1-e)R$. In view of Lemma 7.3 (i), we get $[a, (1-e)L] = 0$. In particular, $\big[a, [(1-e)L, R]\big] = 0$.

Also, $0 = a(1-e)La = a^2(1-e)L$, implying

$$(1-e)a^2\big[(1-e)L, (1-e)R\big] = 0.$$

It follows from Eq.(3) that $(1-e)a^2 = 0$, that is, $b^2 = 0$. By Lemma 3.9 (i), we have $[L, R] = [\widetilde{L}, R]$ and so $\big[a, [(1-e)\widetilde{L}, R]\big] = 0$, implying

$$\big[(1-e)\widetilde{L}, [b, (1-e)R]\big] = 0. \tag{4}$$

Let P be a prime ideal of $(1-e)R$ and let $\overline{(1-e)R} := (1-e)R/P$. By Eq.(4), we have

$$\big[(1-e)\widetilde{L} + P/P, [\overline{b}, \overline{(1-e)R}]\big] = \overline{0}.$$

If $\overline{b} \notin Z(\overline{(1-e)R})$, it follows from Lemma 4.1 that $\overline{x}^2 \subseteq Z(\overline{R})$ for all $x \in (1-e)\widetilde{L}$. Otherwise, $\overline{b} \in Z(\overline{R})$ and so $\overline{b} = \overline{0}$ as $b^2 = 0$. In either case, we have

$$b[x^2, (1-e)R] \subseteq P$$

for all $x \in (1-e)\widetilde{L}$. Since P is arbitrary, the semiprimeness of $(1-e)R$ forces $b[x^2, R] = 0$ for all $x \in \widetilde{L}$. Hence

$$b\sum_{x\in\widetilde{L}} R[x^2, R]R = 0.$$

Thus $b \in \ell_Q\Big(\sum_{x\in\widetilde{L}} R[x^2, R]R\Big) = eQ$. Since $b = (1-e)a$, we get $b = 0$, as desired.

Finally, assume that R is 2-torsion free. Since (i) holds, we have $ex^2 \in C$ for all $x \in \widetilde{L}$. Note that eL is a Lie ideal of the 2-torsion free semiprime ring eR. It follows from Lemma 7.6 that $eL \subseteq C$. This implies that $e[L, R] = 0$ and so $e = 0$ as $\ell_R([L, R]) = 0$. Thus R is L-semiprime. $\square$

The following extends Theorem 6.1 to the general case without the assumption $\ell_R([L, R]) = 0$.

Theorem 7.7. *Let R be a semiprime ring, and let L be a Lie ideal of R. Then there exist orthogonal idempotents $e_1, e_2, e_3 \in C$ with $e_1 + e_2 + e_3 = 1$ such that*
(i) $e_1 L \subseteq C$,
(ii) $e_2 x^2 \in C$ for all $x \in \widetilde{L}$, and
(iii) $e_3 R$ is $e_3 L$-semiprime.
In addition, if R is 2-torsion free, then $(e_1 + e_2)L \subseteq C$.

Proof. Let $Q := Q(R)$. In view of Lemma 7.2, there exists an idempotent $e_1 \in C$ such that

$$\ell_Q([L, R]) = e_1 Q.$$

Clearly, we have $e_1 L \subseteq C$. This proves (i). Let $f := 1 - e_1$. Then fQ is the Martindale symmetric ring of quotients of fR (see [1]). It is clear that $\ell_{fQ}([fL, fR]) = 0$.

In view of Theorem 6.1, there exists an idempotent $e_2 \in fC$ such that $e_2 x^2 \in C$ for all $x \in \widetilde{L}$. This proves (ii). Moreover, $e_3 R$ is $e_3 L$-semiprime, where $e_3 := f - e_2$ and hence (iii) is proved.

Finally, assume that R is 2-torsion free. By Theorem 6.1, $e_2 L \subseteq C$ and hence $(e_1 + e_2)L \subseteq C$, as desired. □

In a prime ring R, a Lie ideal L of R is noncentral iff $\ell_R([L, R]) = 0$. Thus we have the following (see also Theorem 4.5 (i)).

Corollary 7.8. *Let R be a prime ring of characteristic $\neq 2$, and let L be a noncentral Lie ideal of R. Then R is L-semiprime.*

Remark 7.9. There exists a prime ring R of characteristic 2 and a Lie ideal L of R such that $\ell_R([L, R]) = 0$ but R is not L-semiprime (see Remark 6.2 (i)).

Proof of Theorem 6.4.

"$\Longrightarrow$": Clearly, $[B, R] = [B^+, R]$. Let $a[B^+, R]a = 0$, where $a \in R$. The aim is to prove $a = 0$.

Let $e \in B$ and $x \in R$. Then $ex(1-e) = [e, ex(1-e)] \in [B, R]$ and so $aex(1-e)a = 0$. Hence $aeR(1-e)a = 0$ and the semiprimeness of R implies $(1-e)ae = 0$. Similarly, we have $ea(1-e) = 0$ and so $[e, a] = 0$. That is, $[a, B] = 0$ and so $[a, B^+] = 0$. Since B^+ is a Lie ideal of R, we get $\big[a, [B^+, R]\big] = 0$ and hence

$$\big[B^+, [a, R]\big] = 0.$$

Let $e \in B$, and let P be a prime ideal of R. Then $\big[\overline{e}, [\overline{a}, \overline{R}]\big] = \overline{0}$, where $\overline{R} := R/P$.

If $\overline{e} = \overline{e}^2 \notin Z(\overline{R})$, by Lemma 4.1 we get $\overline{a} \in Z(\overline{R})$ and so $[a, R] \subseteq P$. Otherwise, we have $\overline{e} \in Z(\overline{R})$. That is, $[e, R] \subseteq P$. In either case, we conclude that $[a, R][B, R] \subseteq P$. Since P is an arbitrary prime ideal of R, the semiprimeness of R implies $[a, R][B, R] = 0$. Since $\ell_R([B, R]) = 0$, we get $[a, R] = 0$ and so $a \in Z(R)$.

By the fact that $a[B, R]a = 0$, we get $a^2[B, R] = 0$ and so $a^2 = 0$. Since the center of a semiprime ring is reduced, we have $a = 0$, as desired.

"$\Longleftarrow$": Suppose not. Then $\ell_R([B, R]) \neq 0$. Choose a nonzero $a \in \ell_R([B, R])$. Then $a[B, R]a = 0$, a contradiction. □

The following extends Theorem 6.4 to the general case without the assumption $\ell_R([B, R]) = 0$.

Theorem 7.10. *Let R be a semiprime ring, and let B be a subset of $Id(R)$ such that B^+ is a Lie ideal of R. Then there exists an idempotent $e \in C$ such that*

(i) $eB^+ \subseteq C$, *and*
(ii) $(1-e)R$ *is* $(1-e)[B,R]$*-semiprime.*

Proof. Let $Q := Q(R)$. It follows from Lemma 7.2 that

$$\ell_Q([B,R]) = \ell_Q(Q[B^+,R]Q) = eQ$$

for some $e = e^2 \in C$. Then $[eB^+,R] = 0$, implying $eB^+ \subseteq C$ and this proves (i).

Note that $(1-e)B \subseteq Id((1-e)R)$ and $(1-e)B^+$ is a Lie ideal of the semiprime ring $(1-e)R$. Let $a \in (1-e)R$ be such that $a[(1-e)B^+,(1-e)R] = 0$. Then $a[B^+,R] = 0$ and so $a \in eQ$. Hence $a = 0$ follows. In view of Theorem 6.4, $(1-e)R$ is $(1-e)[B,R]$-semiprime and this proves (ii). □

The following complements Proposition 3.2 (see Problem 3.3).

Theorem 7.11. *Let* R *be a semiprime ring. Then* $\ell_R([E(R),R]) = 0$ *iff* R *is* $[E(R),R]$*-semiprime.*

Corollary 7.12. If R is a semiprime ring such that $[E(R),R]$ contains a unit of R, then R is $[E(R),R]$-semiprime.

8. Regular rings

When dealing with idempotent semiprime rings, it is natural to consider rings having many idempotents. Since regular rings also have many idempotents, a comparison is in order. Note that regular rings are unit-semiprime (see [6]).

Theorem 8.1. *Let* R *be a semiprime ring. Suppose that, given any prime ideal* P *of* R*, either* R/P *is a domain or there exists an idempotent* $e \in R$ *such that* $\overline{e} \notin Z(\overline{R})$*, where* $\overline{R} := R/P$*. Then* R *is idempotent semiprime.*

Proof. We let $E := E(R)$. Let $aEa = 0$,where $a \in R$. The aim is to prove $a = 0$.

Let P be a prime ideal of R. We have $\overline{a}\overline{E}\,\overline{a} = \overline{0}$ in $\overline{R} := R/P$. We divide the argument into two cases.

Case 1: $\overline{R}$ is a domain. Then $\overline{a}\overline{E}\,\overline{a} = \overline{0}$ and so $a \in P$.

Case 2: There exists an idempotent $e \in R$ such that $\overline{e} \notin Z(\overline{R})$. Then $\overline{a}\overline{E}\,\overline{a} = \overline{0}$ in $\overline{R}$. Note that $\overline{E}$ is also a Lie ideal of the prime ring $\overline{R}$. By Lemma 3.12 we have

$$[\overline{E},\overline{R}] = \overline{[E,R]} = \overline{[E,E]} = [\overline{E},\overline{E}].$$

Thus $[\overline{E},\overline{E}] \neq 0$ since $\overline{e} \notin Z(\overline{R})$. By Theorem 3.6 (ii), the prime ring $\overline{R}$ is $\overline{E}$-prime. Thus $\overline{a} = 0$ in $\overline{R}$ and so $a \in P$.

In either case, we have $a \in P$. Since P is arbitrary, we get $a = 0$. Thus R is idempotent semiprime. □

Theorem 8.2. *Every regular ring is idempotent semiprime.*

Proof. Let R be a regular ring and let P be a prime ideal of R. Let $\overline{R} := R/P$. Suppose that $\overline{e} \in Z(\overline{R})$ for any idempotent $e \in R$.

Let $a \in R$. Since R is regular, $aba = a$ for some $b \in R$ and so ab is an idempotent of R. Hence $\overline{ab} \in Z(\overline{R})$ and so $\overline{b}\overline{a}^2 = \overline{a}$. Therefore $\overline{R}$ is a reduced ring. By the primeness of $\overline{R}$, $\overline{R}$ is a domain.

The above shows that every regular ring satisfies the assumptions of Theorem 8.1. Thus Theorem 8.1 implies that R is idempotent semiprime. □

Example 8.3. (i) There exists a regular ring R which is $E(R)$-semiprime but is not $[E(R), R]$-semiprime. For instance, let $R := R_1 \oplus R_2$, where $R_1 = M_n(A)$ with A a regular ring, $n > 1$, and R_2 is a division ring. Clearly, R is a regular ring. Then

$$\ell_R([E(R), R]) = \ell_{R_1}([E(R_1), R_1]) \oplus \ell_{R_2}([E(R_2), R_2]) = 0 \oplus R_2 \neq 0.$$

In view of Theorem 7.11, R is not $[E(R), R]$-semiprime. However, by Theorem 8.2, R is $E(R)$-semiprime.

(ii) In the ring R given in (i), it is easy to check that, given any prime ideal P of R, either R/P is a domain or there exists an idempotent $e \in R$ such that $\overline{e} \notin Z(\overline{R})$, where $\overline{R} := R/P$. Thus, in Theorem 8.1, we cannot conclude that R is $[E(R), R]$-semiprime.

9. Subdirect products

We always assume that, given any ring R, there exists a subset $X(R)$ associated with R. For instance, let $X(R)$ be $E(R)$, $U(R)$, $[E(R), R]$, $[R, R]$ etc. The following is clear.

Theorem 9.1. *Given rings R_β, $\beta \in J$, an index set, if $X(\prod_{\beta \in J} R_\beta) = \prod_{\beta \in J} X(R_\beta)$, then $\prod_{\beta \in J} R_\beta$ is $X(\prod_{\beta \in J} R_\beta)$-semiprime iff R_β is $X(R_\beta)$-semiprime for all $1\beta \in J$.*

An idempotent semiprime ring R just means that it is $Id(R)$-semiprime. The following is a direct consequence of of Theorem 9.1 by applying the property $Id(\prod_\beta R_\beta) = \prod_\beta Id(R_\beta)$ for rings R_β.

Proposition 9.2. *A direct product of rings is idempotent semiprime iff each component is idempotent semiprime.*

Theorem 9.3. *Let R be a semiprime ring. Suppose that, given any prime ideal P of R, there exists an idempotent $e \in R$ such that $\overline{e} \notin Z(\overline{R})$, where $\overline{R} := R/P$. Then the following hold:*

(i) R is $[E(R), R]$-semiprime;

(ii) $\overline{R}$ is $[E(\overline{R}), \overline{R}]$-prime for any prime ideal P of R, where $\overline{R} := R/P$;

(iii) R is a subdirect product of prime rings R_β, $\beta \in I$, an index set, such that each R_β is $[E(R_\beta), R_\beta]$-prime.

Proof. (i) Let $E := E(R)$. By Theorem 7.11, it suffices to show that $\ell_R([E, R]) = 0$. Suppose not, that is, $\ell_R([E, R]) \neq 0$. There exists a nonzero $a \in R$ such that $a[E, R] = 0$. In particular, $a[E, R^2] = 0$ and so $aR[E, R] = 0$. Since R is a semiprime ring, $a \notin P$ for some prime ideal P of R. Then $aR[E, R] \subseteq P$, implying $[E, R] \subseteq P$. Hence $\overline{e} \in Z(\overline{R})$ for any idempotent $e \in R$, where $\overline{R} := R/P$, a contradiction.

(ii) In view of Theorem 7.11, it suffices to show that $\ell_{\overline{R}}([E(\overline{R}), \overline{R}]) = \overline{0}$. Suppose that $\overline{a}[E(\overline{R}), \overline{R}] = \overline{0}$, where $\overline{a} \in \overline{R}$. Since $\overline{E(R)} \subseteq E(\overline{R})$, we get $\overline{a}[\overline{E(R)}, \overline{R}] = \overline{0}$ and so $\overline{a}\overline{R}[\overline{E(R)}, \overline{R}] = \overline{0}$. Since, by assumption, $[\overline{E(R)}, \overline{R}] \neq \overline{0}$, it follows from the primeness of $\overline{R}$ that $\overline{a} = \overline{0}$, as desired. Thus $\overline{R}$ is $[E(\overline{R}), \overline{R}]$-semiprime. The primeness of $\overline{R}$ implies that $\overline{R}$ is $[E(\overline{R}), \overline{R}]$-prime.

(iii) Since every semiprime ring is a subdirect product of prime rings, (iii) follows directly from (ii). □

Notice that Theorem 3.15 is also an immediate consequence of Theorem 9.3 (i).

Theorem 9.4. *Let R be a semiprime ring. Suppose that $\overline{E(R)} = E(\overline{R})$ for any prime homomorphic image $\overline{R}$ of R. If R is a subdirect product of $[E(R_\beta), R_\beta]$-prime rings R_β, $\beta \in I$, an index set, then R itself is $[E(R), R]$-semiprime.*

Proof. Write $R_\beta = R/P_\beta$, where P_β is a prime ideal of R, for each $\beta \in I$. By Theorem 7.11, it suffices to prove that $\ell_R([E(R), R]) = 0$. Otherwise, $a[E(R), R] = 0$ for some nonzero $a \in R$. Since $\bigcap_{\beta\in I} P_\beta = 0$, there exists $\beta \in I$ such that $\overline{a} \neq 0$ in $R_\beta = R/P_\beta$.

By assumption, we have $E(R_\beta) = \overline{E(R)}$. So $\overline{a}[E(R_\beta), R_\beta] = 0$ and hence $\ell_{R_\beta}([E(R_\beta), R_\beta]) \neq 0$. Theorem 7.11 implies that R_β is not $[E(R_\beta), R_\beta]$-prime, a contradiction. □

Motivated by Theorem 9.4, it is natural to ask the following

Problem 9.5. *Characterize semiprime rings R satisfying the property that $\overline{E(R)} = E(\overline{R})$ for any prime homomorphic image $\overline{R}$ of R.*

Recall that if A is an additive subgroup of a ring R, we say that idempotents can be lifted modulo A if, given $x \in R$ with $x - x^2 \in A$, there exists $e = e^2 \in R$ such that $e - x \in A$.

A ring R is called *suitable* (or *exchange* [20]) if, given any $x \in R$, there exists $e = e^2 \in R$ with $e - x \in R(x - x^2)$. Nicholson proved that a ring is suitable iff idempotents can be lifted modulo every left ideal (see [20, Corollary 1.3]). Hence we have

Proposition 9.6. *Let R be a suitable ring. Then $\overline{E(R)} = E(\overline{R})$ for any prime homomorphic image $\overline{R}$ of R.*

The class of suitable rings is large: every homomorphic image of a suitable ring, semiregular rings, clean rings and many others (see [20]).

Theorem 9.7. *Let R be a semiprime suitable ring. Assume that R/P contains a nontrivial idempotent for any prime ideal P of R. Then R is $[E(R), R]$-semiprime.*

Proof. Since R is a semiprime ring, R is a subdirect product of prime rings R_β, $\beta \in I$, an index set. By the fact that R is suitable, it follows from Proposition 9.6 that $[E(R), R] + P_\beta/P_\beta = [E(R_\beta), R_\beta]$ for any $\beta \in I$. In view of Theorem 3.13, every R_β is $[E(R_\beta), R_\beta]$-prime. Hence, by Theorem 9.4, R is $[E(R), R]$-semiprime. □

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