UNIT-REGULAR MATRICES OVER RINGS

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Abstract. First, unit-regular matrices are characterized up to similarity over any ID ring, that is, a ring in which every idempotent matrix is similar to a diagonal matrix. Second, the structural properties of unit-regular matrices are investigated across several classes of rings, including certain commutative rings.

1. Introduction

In [4] (1968), with further development in [5] (1976), Gertrude Ehrlich introduced and studied a special class of von Neumann regular rings, which she termed unit-regular rings. An element a of a unital ring R is called *unit-regular* if there exists a unit u in R such that $a = aua$. Denote by $U(R)$ the set of all the units of R and by $ureg(R)$ the set of all unit-regular elements in R. A unital ring is called unit-regular if all its elements are unit-regular.

The relationship between unit-regular rings and their matrix rings is well established. Henriksen [7] (1973) demonstrated that matrix rings over unit-regular rings are also unit-regular. Conversely, if the matrix ring $\mathbb{M}_n(R)$ is unit-regular for some $n \geq 2$, then R is unit-regular. This result was attributed to Kaplansky in the 1971 paper by Hartwig and Luh [6]. (In fact, if R is any unit-regular ring, then any corner ring eRe is also unit-regular; see Ex. 21.9 in [10]).

A natural question, not addressed in Gertrude Ehrlich's papers, is to characterize the unit-regular matrices over the broadest possible classes of rings. As far as we could determine, aside from one very specific case, no prior work has explicitly stated or proved such characterizations. The exception can be found in [9] (2004), which examines 2×2 matrices with zero second row over commutative (unital) rings. Specifically, a matrix $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is unit-regular if and only if $(a, b) = e(a', b')$ for some idempotent e and some unimodular row (a', b').

In the second section of this work, we describe, up to similarity, the unit-regular matrices of any size over an ID ring. Following Steger [12] (1966), a ring R is called an *ID* ring if every idempotent matrix over R is similar to a diagonal matrix. Examples of ID rings include division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

In the third section, characterizations are provided for unit-regular matrices over commutative ID rings. The final section presents a characterization over pre-Scheier domains.

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To elaborate, an n-tuple $(a_1, ..., a_n)$ forms a *unimodular row* if $a_1R + ... + a_nR = R$, or equivalently, if there exist $x_1, ..., x_n \in R$ such that $a_1x_1 + ... + a_nx_n = 1$. We denote this by $(a_1, ..., a_n) \in Um(R^n)$. A 2×2 matrix $A = [a_{ij}]$ is said to have *unimodular entries* if $(a_{11}, a_{12}, a_{21}, a_{22}) \in Um(R^4)$.

The *inner rank* of an $n \times n$ matrix A over a ring is defined as the smallest integer m such that A can be expressed as the product of an $n \times m$ matrix and an $m \times n$ matrix. For instance, over a division ring, this notion aligns with the standard definition of rank. A square matrix is called *full* if its inner rank equals its order and *non-full* otherwise.

Throughout this paper, the term *regular*, when referring to elements or rings, will denote von Neumann regular. Two elements a, b in a ring R are *equivalent* if there exist units p, q such that $b = paq$.

2. The characterization

The following well-known facts are recalled for the reader's convenience.

Lemma 2.1. ***a)*** *If a is regular and $a = axa$ then both ax and xa are idempotents. The converses fail.*

b) *If a is unit-regular and $a = aua$ with some unit u then both au and ua are idempotents. Both converses hold, i.e., if au (or ua) is idempotent then a is unit-regular.*

c) *An element a is unit-regular iff $a = eu$ (or $a = ue$) with $e^2 = e \in R$ and unit $u \in R$.*

d) *If R is a connected ring (i.e., has only the trivial idempotents 0 and 1) then the only unit-regular elements of R are 0 and the units.*

*(**e**) Unit-regular elements are invariant under equivalences.*

Proof. a) In $\mathbb{Z}_{12}$, $2 \cdot 2 = 4 = 4^2$ but 2 is not regular in $\mathbb{Z}_{12}$.

e) If $a = aua$ then $(paq)(q^{-1}up^{-1})(paq) = paq$, for any units p, q and u. □

Since our characterization of unit-regular matrices is up to similarity, we adopt the notation $\sim$ to denote the conjugation (binary) relation in a ring R. More precisely, for elements $a, b \in R$, we write $a \overset{v}{\sim} b$ if there exists a unit $v \in R$ such that $b = vav^{-1}$; in this case, we say that b *is conjugate to* a *via* v. The following straightforward results will be useful.

Lemma 2.2. *If $a \overset{v}{\sim} b$ and $c \overset{v}{\sim} d$ then $ac \overset{v}{\sim} bd$.*

Lemma 2.3. *Suppose $e \overset{v}{\sim} f$ and $a = eu$ for a unit $u \in R$. There exists a unit $w \in R$ such that $a \overset{v}{\sim} b = fw$.*

Proof. Just a special case of the previous lemma. If $f = vev^{-1}$ then $w = vuv^{-1}$. Indeed $b = fw = vev^{-1}vuv^{-1} = veuv^{-1} = vav^{-1}$. □

In particular, if $a = eu$ is unit-regular and f is an idempotent conjugate to e via v, there is a unit w such that the unit-regular $b = fw$ is conjugate to a via v. Recall that, for matrices, conjugation is traditionally referred to as similarity.

This completes the necessary groundwork to state and prove the characterization. For the reader's convenience, we present the result separately for 2×2 matrices.

First of all, the case of a 1×1 matrix $A = [a]$ is obvious (by Lemma 2.1, (c)): A is unit-regular iff $a = eu$ for some idempotent e and unit u.

Proposition 2.4. *Up to similarity, the unit-regular* 2×2 *matrices over an ID ring are of form* $e\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ *or* $e\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}+e'\begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix}$ *with idempotents* e, e' *and invertible* $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

Proof. When describing, up to similarity, the unit-regular matrices $A = EU$ over an ID ring, by Lemma 2.3, we can suppose E is diagonal (with idempotent entries). Up to similarity these are $\begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} e & 0 \\ 0 & e' \end{bmatrix}$. Thus, if $U=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is invertible, these are $e\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ where $(x, y) \in Um(R^2)$ or else $\begin{bmatrix} e & 0 \\ 0 & e' \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}=\begin{bmatrix} ex & ey \\ e'z & e'w \end{bmatrix}$ with invertible $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ (and so $(x, y),(z, w) \in Um(R^2)$). In the second case, the product decomposes as $e\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}+e'\begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix}$. □

Remarks. 1) If $U=\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is invertible then $(x, y),(z, w) \in Um(R^2)$, but the converse may fail (even if the base ring is supposed commutative).

2) However, for the first type $e\begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$, it suffices to assume $(x, y) \in Um(R^2)$, since every unimodular 2-row is completable to an invertible 2×2 matrix. Indeed, if $xr - ys = 1$ then $\begin{bmatrix} x & y \\ s & r \end{bmatrix}$ is invertible.

3) Note that $\begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix}=U_2\begin{bmatrix} w & z \\ 0 & 0 \end{bmatrix}U_2$ where $U_2=U_2^{-1}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, is also a similarity.

The $n \times n$ case is analogous.

Theorem 2.5. *Up to similarity, the* $n\times n$ *unit-regular matrices over an ID ring are of form* $e_1\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ *or* $e_1\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}+e_2\begin{bmatrix} 0 & \cdots & 0 \\ u_{21} & \cdots & u_{2n} \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ *or*

... *or* $e_1\begin{bmatrix} u_{11} & \cdots & u_{1n} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}+e_2\begin{bmatrix} 0 & \cdots & 0 \\ u_{21} & \cdots & u_{2n} \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}+...+e_n\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ u_{n1} & \cdots & u_{nn} \end{bmatrix}$

for some idempotents $e_1, e_2, ..., e_n$ *and an invertible matrix* $U = [u_{ij}]_{1\le i,j\le n}$.

In a more compact form, a matrix U remains invertible after any permutation of rows or columns whence any subsum $\sum_{i=j}^{k} e_iU_i$ is unit-regular where $1 \le j \le k-1 \le n-1$ and U_i denotes the matrix whose i-row is the i-row of U and all the other entries are 0.

Remark. The matrices above, which have only one nonzero row, are similar to the matrices which have only the first nonzero row. To simplify the writing we give the details in the $n = 3$ case.

Denoting $U_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (for which $U_3^2 = I_3$) note the similarity

$$U_3 \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} U_3 = \begin{bmatrix} u_{33} & u_{32} & u_{31} \\ u_{23} & u_{22} & u_{21} \\ u_{13} & u_{12} & u_{11} \end{bmatrix}.$$

As special case of this similarity, $U_3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ u_{31} & u_{32} & u_{33} \end{bmatrix} U_3 = \begin{bmatrix} u_{33} & u_{32} & u_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Also $V_3 \begin{bmatrix} 0 & 0 & 0 \\ u_{21} & u_{22} & u_{23} \\ 0 & 0 & 0 \end{bmatrix} V_3^2 = \begin{bmatrix} u_{22} & u_{23} & u_{21} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for $V_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ (for which $V_3^3 = I_3$).

3. Over commutative rings

In this section, as indicated by the title, the ring R is assumed to be (unital and) commutative.

The role of idempotents in the general description of unit-regular matrices (not restricted to those with unimodular entries or over connected rings) was foreseeable, due to the result on matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ mentioned in the Introduction (see [9]), as well as the existing characterization of regular matrices given by the theorem below, proved in [11].

For any matrix $A \in \mathbb{M}_n(R)$, let $D_i(A)$ $(1 \leq i \leq n)$ denote the *i-th determinantal ideal* of A, that is, the ideal in R generated by the $i \times i$ minors of A (see [1]). We have a descending sequence of ideals

$$D_0(A) \supseteq D_1(A) \supseteq ... \supseteq D_n(A) = \det(A)R \supseteq (0),$$

where, by convention, $D_0(A) = R$.

Theorem 3.1. *A matrix $A = (a_{ij}) \in \mathbb{M}_n(R)$ over a commutative ring R is regular iff each determinantal ideal $D_i(A)$ $(0 \leq i \leq n)$ is idempotent (or equivalently, each $D_i(A)$ is generated by an idempotent in R).*

Note that if R is a connected ring, the theorem shows that A *is regular iff each* $D_i(A)$ *is either* (0) *or* R.

Suppose A is unit-regular and $\det(A)$ *is not a zero divisor*. Then $A = AUA$ for some invertible matrix U and from $\det(A)\det(U)\det(A) = \det(A)$ we get $\det(AU) = \det(UA) = 1$. Hence both AU, UA are invertible and so is A. Therefore

Proposition 3.2. *Over any integral domain, only invertible or zero determinant matrices may be unit-regular.*

For pre-Schreier integral domains the converse also holds (see Theorem 4.2).

Also note that

Proposition 3.3. *If A is unit-regular then* $\det(A)$ *is unit-regular too. The converse may fail.*

Proof. We just provide examples regarding the failure of the converse for 2×2 matrices.

Over any commutative ring R, suppose $0 \neq a$ is such that a^2 is idempotent and a^2 does not divide a. Consider aI_2 in $\mathbb{M}_2(R)$. Then $\det(aI_2) = a^2$ is idempotent (so trivially unit-regular) but aI_2 is not even regular. Indeed, $(aI_2)X(aI_2) = aI_2$ amounts to $a^2X = aI_2$ and further to $a^2x = a$ for some x, with no solution. For example take $aI_2 = 2I_2$ in $\mathbb{M}_2(\mathbb{Z}_{12})$. Then $4 = 2^2$ is idempotent but $4x = 2$ has no solutions in $\mathbb{Z}_{12}$.

Alternatively, let $0 \neq a$ with $a^2 = 0$. Then $\det(aI_2) = a^2 = 0$ (is trivially unit-regular) but again aI_2 is not even regular. Indeed, $(aI_2)X(aI_2) = aI_2$ amounts to $a^2X = 0_2 \neq aI_2$. For example take $aI_2 = 6I_2$ in $\mathbb{M}_2(\mathbb{Z}_{12})$. Then $6^2 = 0$ and $(6I_2)X(6I_2) = 0_2 \neq 6I_2$.

Less trivially, $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is not (even) regular over $\mathbb{Z}_{12}$, as $AXA = A$ amounts again to $4x = 2$ (with no solutions), but here $\det(A) = 2$ is a zero divisor that is not idempotent nor zero-square. □

It follows that - excluding the invertible matrices - we only need to describe the unit-regular matrices whose determinant is a zero divisor. In particular, this includes unit-regular matrices with idempotent determinants, and more specifically, those with determinant zero.

As already noted in [11] (in the context of regular matrices), the classification of unit-regular matrices can be reduced to the case where the determinant is zero.

For $n = 2$, Theorem 3.1 can be stated as follows:

Theorem 3.4. *A matrix $A = (a_{ij}) \in \mathbb{M}_2(R)$ over a commutative ring R is regular iff there exist idempotents $e, e' \in R$ such that $D_1(A) = eR$ and $D_2(A) = (\det A)R = e'R$.*

The reduction is as follows.

If a matrix $A \in \mathbb{M}_2(R)$ has $D_1(A) = eR$ and $D_2(A) = (\det A)R = e'R$, where e, e' are idempotents, then R splits into $e'R \times (1 - e')R$, and in the component $e'R$, the projection $e'A$ of A is *invertible* (and so also *unit-regular*). Thus, for unit-regular 2×2 matrices, it suffices to analyze the projection of A in the other component $(1 - e')R$, which has determinant zero.

Note that $\det(A) = 0$ implies $\det(E) = 0$, whenever $A = EU$ is unit-regular with idempotent E and invertible U. By the characterization above, (say) in the 2×2 case, this can only happen, up to similarity, when $A \sim e \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix}$ with $(x, y) \in Um(R^2)$ or $A \sim \begin{bmatrix} e & 0 \\ 0 & e' \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ with orthogonal (in particular, complementary) idempotents e, e' and invertible matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$.

For completeness, we collect here some basic results concerning unimodular rows and invertible matrices.

Lemma 3.5. *A non-full matrix $A = [a_{ij}] = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$ has unimodular entries iff both $(a, b), (s, t) \in Um(R^2)$.*

Proof. "$\Leftarrow$" If $a\alpha+b\beta=1=s\sigma+t\tau$ then we get $a_{11}\alpha\sigma+a_{12}\beta\sigma+a_{21}\alpha\tau+a_{22}\beta\tau=(a\alpha+b\beta)(s\sigma+t\tau)=1$.

"$\Rightarrow$" If there exist $(as)x+(bs)y+(at)z+(bt)w=1$ we can choose $\alpha=sx+tz$, $\beta=sy+tw$ and similarly $\sigma=ax+by$, $\tau=az+bw$. □

Lemma 3.6. *If* $U=\begin{bmatrix} x & z \\ y & w \end{bmatrix}\in GL_2(R)$ *then*

(i) $(x,z),(y,w)\in Um(R^2)$*;*

(ii) for any $r\in R$*,* $(x,z)+r(z,w)\in Um(R^2)$*.*

Proof. (i) Suppose $xR+zR=\delta R$ with $\delta\notin U(R)$. Since $x,z\in\delta R$ we have $x=\delta x'$, $z=\delta z'$ and so $\det(U)=\delta r$ for some r. Then since $\delta\notin U(R)$ so is $\det(U)\notin U(R)$, and $U\notin GL_2(R)$.

(ii) This corresponds to an elementary transformation: $r\mathrm{row}_2(U)+\mathrm{row}_1(U)$, which does not change the determinant, but only the first row. Now we apply (i). □

A symmetric statement holds for columns.

Lemma 3.7. *If* $\begin{bmatrix} a & b \end{bmatrix}$ *and* $\begin{bmatrix} x \\ z \end{bmatrix}$ *are unimodular there exists a matrix* U *with* $\det(U)=1$ *such that* $\begin{bmatrix} a & b \end{bmatrix}U\begin{bmatrix} x \\ z \end{bmatrix}=[1]$.

Proof. Write $aa'+bb'=1=xx'+zz'=1$, and consider the matrices A and X in $SL_2(R)$ such that $A=\begin{bmatrix} a' & -b \\ b' & a \end{bmatrix}$ and $X=\begin{bmatrix} x' & z' \\ -z & x \end{bmatrix}$. If we take $U=AX$, then $\begin{bmatrix} a & b \end{bmatrix}U\begin{bmatrix} x \\ z \end{bmatrix}=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}=[1]$ and $\det(U)=1$. □

We have shown that over an ID ring, a unit-regular 2×2 matrix is similar either to a matrix of form $e\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ for some idempotent e and unimodular (a,b) (*type 1*), or to a matrix of form $\begin{bmatrix} e & 0 \\ 0 & e' \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some idempotents e, e' and invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (*type 2*). Since our analysis is restricted to zero determinant matrices, in the type 2 we assume e and e' are orthogonal idempotents.

Over commutative ID rings, we first characterize *the type 1 unit-regular matrices.* Clearly, all such matrices have zero determinant.

Theorem 3.8. *Over a commutative ID ring* R*, a zero determinant* 2×2 *matrix* A *is unit-regular of type 1 iff there exists an idempotent* $e\in R$ *such that* $A=eBV$ *with*

(i) B *is unimodular non-full (i.e., a column-row product),*

(ii) V *is invertible,*

(iii) $\begin{bmatrix} x & y \\ z & w \end{bmatrix}=V^{-1}$ *and* $B=\begin{bmatrix} x \\ z \end{bmatrix}\begin{bmatrix} a & b \end{bmatrix}$ *for a unimodular row* $\begin{bmatrix} a & b \end{bmatrix}$*.*

Proof. In one direction, reversing the similarities, the unit-regular matrices of this type are just products $A = eU \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} U^{-1}$ with invertible U. If $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, then by computation $A = e \begin{bmatrix} x \\ z \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} U^{-1}$. Thus $V = U^{-1}$ gives (iii).

Conversely, we are searching for an invertible matrix U such that $A = AUA$, that is, $eBV = eBVUeBV$. It suffices to find U such that $B = BVUB$, by right multiplication with V^{-1} and $e^2 = e$. As already done in Lemma 3.7, suppose $aa' + bb' = 1 = xx' + zz'$ and consider $M = \begin{bmatrix} a' & -b \\ b' & a \end{bmatrix}$, $N = \begin{bmatrix} x' & z' \\ -z & x \end{bmatrix}$ and $W = MN$. It is readily checked that $\begin{bmatrix} a & b \end{bmatrix} W \begin{bmatrix} x \\ z \end{bmatrix} = [1]$ so, if we choose $U = V^{-1}W$, we get $B = BWB = BVUB$, as desired. □

Examples. 1) For the matrices considered in [11], $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, we have the decomposition $e \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} a' & b' \end{bmatrix} I_2$ with unimodular $\begin{bmatrix} a' & b' \end{bmatrix}$.

2) For zero determinant idempotent 2×2 matrices E, by Cayley-Hamilton's theorem, we have $E = Tr(E)E$ and since similar matrices have the same trace, in the type 1 case, $Tr(E) = e$, an idempotent. Thus, $eE = E = \begin{bmatrix} a & b \\ c & e-a \end{bmatrix}$ with $a(e-a) = bc$ and $a = ea$, $b = eb$, $c = ec$. Hence $E = e \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $a(1-a) = bc$.

Further note that since $E' = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is conjugate to a diagonal matrix (the base ring is supposed ID commutative) there exist $x, y, x_0, y_0 \in R$ such that $a = xy$, $c = xx_0$ and $b = yy_0$ (see Proposition 18, [3]). This is because assuming E is conjugate to $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, it follows that R^2 has a basis $\{(y, x_0)^T, (-y_0, x)^T\}$ such that $EU = E \begin{bmatrix} y & -y_0 \\ x_0 & x \end{bmatrix} = \begin{bmatrix} y & 0 \\ x_0 & 0 \end{bmatrix}$, where we may assume that $xy + x_0y_0 = 1$. Therefore $E = \begin{bmatrix} y & 0 \\ x_0 & 0 \end{bmatrix} \begin{bmatrix} x & y_0 \\ -x_0 & y \end{bmatrix}$. This gives right away $a = xy$,$b = yy_0$, $c = xx_0$, as desired. Finally $E = e \begin{bmatrix} y \\ x_0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} U^{-1} = e \begin{bmatrix} y \\ x_0 \end{bmatrix} \begin{bmatrix} x & y_0 \end{bmatrix} I_2$ with invertible $U = \begin{bmatrix} y & -y_0 \\ x_0 & x \end{bmatrix}$.

We mention that if R is a GCD domain (greatest common divisors exist) then $x = \gcd(a, c)$ and if $a = xy$, $c = xx_0$ with $\gcd(y, x_0) = 1$ then $y \mid b$ and so $b = yy_0$ for some y_0.

Next, we characterize the type 2 unit-regular matrices. Since the idempotents involved are orthogonal, these matrices also have zero determinant. To proceed, we require the following key lemma.

Lemma 3.9. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ *be invertible matrices and* $B = \begin{bmatrix} x \\ z \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$, $B' = \begin{bmatrix} y \\ w \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix}$. *There exists an invertible matrix* U *such that* $B = BUB$ *and* $B' = B'UB'$.

Proof. Since rows and columns of invertible matrices are unimodular, there exist a', b', x', z' such that $xx' + zz' = 1 = aa' + bb'$. Multiplying on left with $\begin{bmatrix} x' & z' \end{bmatrix}$ and on right with $\begin{bmatrix} a' \\ b' \end{bmatrix}$, the first equality reduces to $[1] = \begin{bmatrix} a & b \end{bmatrix} U \begin{bmatrix} x \\ z \end{bmatrix}$. Similarly, the second equality reduces to $[1] = \begin{bmatrix} c & d \end{bmatrix} U \begin{bmatrix} y \\ w \end{bmatrix}$. Now note that for any matrix U, the (1,1) entry of the product AUX is $\begin{bmatrix} a & b \end{bmatrix} U \begin{bmatrix} x \\ z \end{bmatrix}$ and the (2,2) entry is $[1] = \begin{bmatrix} c & d \end{bmatrix} U \begin{bmatrix} y \\ w \end{bmatrix}$. Therefore we can choose $U = A^{-1}X^{-1}$, as $AA^{-1}X^{-1}X = I_2$ has the required diagonal entries. □

Remark. We can choose $U = A^{-1} \begin{bmatrix} 1 & -\det(A)\det(B) + 1 \\ 1 & 1 \end{bmatrix} B^{-1} \in SL_2(R)$.

Here is the characterization for type 2 unit-regular matrices.

Theorem 3.10. *Over a commutative ID ring* R*, a zero determinant* 2×2 *matrix* A *is unit-regular of type 2 iff there exist two orthogonal idempotents* e, e' *such that* $A = (eB + e'B')V$ *where*

(i) B, B' *are unimodular non-full matrices,*

(ii) V *is an invertible matrix,*

(iii) if $B = \begin{bmatrix} x \\ z \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$, $B' = \begin{bmatrix} y \\ w \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix}$ *then* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ *are invertible, and*

(iv) $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = V^{-1}$.

Proof. In one direction, recall that up to similarity, these matrices are of form $e \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} + e' \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$ with invertible $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Reversing the similarity, the unit-regular matrices of this type are $A = eU \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} U^{-1} + e'U \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} U^{-1}$, with invertible U.

Hence, if $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ we get $A = (e \begin{bmatrix} x \\ z \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} + e' \begin{bmatrix} y \\ w \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix})U^{-1}$ with orthogonal idempotents e, e' and invertible $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From $V = U^{-1}$ we obtain (iv).

Conversely, we are searching for an invertible matrix U such that $A = AUA$, that is, $(eB + e'B')V = (eB + e'B')VU(eB + e'B')V$. Equivalently, $eB + e'B' = (eB + e'B')VU(eB + e'B')$ and since $ee' = 0$, $eB + e'B' = eBVUeB + e'B'VUe'B'$. It suffices to find an invertible matrix W such that $B = BWB$ and $B' = B'WB'$. This is done as in Lemma 3.9. □

Example. For zero determinant idempotent matrices, by Cayley-Hamilton theorem we know that $E = Tr(E)E$ and since similar matrices have the same trace, in the type 2 case, $Tr(E) = e + e'$, is also an idempotent, as e and e' are orthogonal. Thus, $E = \begin{bmatrix} e+x & y \\ z & e'-x \end{bmatrix}$ with $(e+x)(e'-x) = yz$. Since $(e+e')E = E$ we get $x = (e+e')x$, $y = (e+e')y$, $z = (e+e')$ and so $E = (e+e')E = eE + e'E$. A decomposition is provided as in example 2, after Theorem 3.8 (since again the base ring is supposed to be ID commutative).

We finally characterize the zero determinant 2×2 matrices whose entries form a unimodular row. In the proof we use, only for 2×2 matrices, the equality

$$\det(A+B) = \det(A) + \det(B) + Tr(A)Tr(B) - Tr(AB).$$

Theorem 3.11. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *be a nonzero* 2×2 *matrix with zero determinant over a commutative ring* R. *If its entries form a unimodular row and among the solutions of* $ax + by + cz + dw = 1$, *there is one such that* $\det \begin{bmatrix} x & z \\ y & w \end{bmatrix} = 0$ *or* $\det \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ *is a unit, then* A *is unit-regular. If* R *is connected (in particular, an integral domain), the converse also holds.*

Proof. If among the solutions of $ax + by + cz + dw = 1$ there is one such that $\det \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ is a unit then $X = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$ is a unit inner inverse for A. If there is one solution such that $\det \begin{bmatrix} x & z \\ y & w \end{bmatrix} = 0$, we show that $U = X + adj(A)$ (the adjugate) is a unit inner inverse for A (with $\det(U) = 1$).

Indeed

$$AUA = A(X + adj(A))A = AXA + Aadj(A)A = A + \det(A)A = A$$

and

$$\det(U) = \det(X + adj(A)) =$$
$$\det(X) + \det(adj(A)) + Tr(X)Tr(adj(A)) - Tr(Xadj(A)) =$$
$$0 + 0 + ax + by + cz + dw = 1.$$

Conversely, suppose A is unit-regular. Then it is regular, the entries form a unimodular row (Theorem 3.1, in the connected case) and so $ax+by+cz+dw = 1$ has solutions. Each solution yields an inner inverse for A, that is, $A = AXA$ for $X = \begin{bmatrix} x & z \\ y & w \end{bmatrix}$. Since A is unit-regular, it has a unit inner inverse U such that $A = AUA$. Since U is invertible, $\det(U)$ is a unit. □

4. Over pre-Schreier domains

A commutative ring R is called *pre-Schreier*, if every nonzero element $r \in R$ is *primal*, i.e., if r divides xy, there are $r_1, r_2 \in R$ such that $r = r_1 r_2$, r_1 divides x and r_2 divides y.

Recall from [2] that

Theorem 4.1. *Over a pre-Schreier domain a* 2×2 *matrix has zero determinant iff it is non-full, that is, it admits a column-row decomposition.*

Proof. Indeed, such matrices have (rank 1 and) dependent rows (or columns), that is, have the form $A = [a_{ij}] = \begin{bmatrix} sa & sb \\ ta & tb \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$. □

For pre-Schreier domains, we can prove the following characterization (including a converse for Proposition 3.2).

Theorem 4.2. *Let R be a pre-Schreier (commutative) domain. A nonzero 2×2 matrix with zero determinant is unit-regular iff its entries form a unimodular row.*

Proof. According to Theorem 4.1 and Lemma 3.5, we can start with a column-row product $A = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$ such that both $(a, b), (s, t) \in Um(R^2)$. We have to find an invertible matrix U such that $A = AUA$. Pre-Schreier domains are Bézout, so since $(s, t) \in Um(R^2)$, there are σ, τ such that $s\sigma + t\tau = 1$, that is, $\begin{bmatrix} \sigma & \tau \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = [1]$ and similarly, there exist α, β such that $a\alpha + b\beta = 1$, that is $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = [1]$. Finally $U = \begin{bmatrix} bt + \sigma\alpha & -bs + \tau\alpha \\ -ta + \beta\sigma & sa + \beta\tau \end{bmatrix}$ is the required invertible matrix.

Indeed, $(bt + \sigma\alpha)(sa + \beta\tau) - (-bs + \tau\alpha)(-ta + \beta\sigma) = (a\alpha + b\beta)(s\sigma + t\tau) = 1$ and $\begin{bmatrix} a & b \end{bmatrix} U = \begin{bmatrix} \sigma & \tau \end{bmatrix}$ is readily checked.

Conversely, assume a non-full 2×2 matrix $A = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$ is unit regular. Thus there exists an invertible matrix U such that

$$\begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} U \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}.$$

Again, by Lemma 3.5, we have to show that both $\begin{bmatrix} s \\ t \end{bmatrix}$ and $\begin{bmatrix} a & b \end{bmatrix}$ are unimodular.

By contradiction (and Bézout hypothesis) assume $sR + tR = \delta R$ with $\delta \notin U(R)$ and let $aR + bR = \gamma R$ (with $\gamma \neq 0$). Since $s, t \in \delta R$, we write $s = \delta s'$, $t = \delta t'$ with unimodular $\begin{bmatrix} s' \\ t' \end{bmatrix}$ and similarly $a, b \in \gamma R$ and $a = \gamma a'$, $b = \gamma b'$, so that

$$\delta\gamma \begin{bmatrix} s' \\ t' \end{bmatrix} \begin{bmatrix} a' & b' \end{bmatrix} = \delta^2\gamma^2 \begin{bmatrix} s' \\ t' \end{bmatrix} \begin{bmatrix} a' & b' \end{bmatrix} U \begin{bmatrix} s' \\ t' \end{bmatrix} \begin{bmatrix} a' & b' \end{bmatrix}.$$

If $\begin{bmatrix} p & q \end{bmatrix} \begin{bmatrix} s' \\ t' \end{bmatrix} = [1]$ and $\begin{bmatrix} a' & b' \end{bmatrix} \begin{bmatrix} m \\ n \end{bmatrix} = [1]$, by left multiplication with $\begin{bmatrix} p & q \end{bmatrix}$ and right multiplication with $\begin{bmatrix} m \\ n \end{bmatrix}$ we get

$$\delta\gamma = \delta^2\gamma^2 \begin{bmatrix} a' & b' \end{bmatrix} U \begin{bmatrix} s' \\ t' \end{bmatrix}.$$

Hence $\delta\gamma = \delta^2\gamma^2 r$, for some r, which by cancellation gives $1 = \delta\gamma r$ so $\delta \in U(R)$, a contradiction. □

Remarks. 1) As the reader may have obseved, the invertibility of U was not used in the proof of the converse. Therefore, *if a non-full matrix is regular over a pre-Schreier domain, its entries must form a unimodular row.*

2) Since for given $(a,b),(s,t) \in Um(R^2)$, there are multiple ways to choose unimodular pairs $(\alpha,\beta),(\sigma,\tau) \in Um(R^2)$, the above procedure yields many distinct invertible inner inverses for $A = \begin{bmatrix} s \\ t \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix}$.

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