# STRONGLY CLEAN MATRICES THAT ARE (NOT) UNIQUELY STRONGLY CLEAN

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Abstract. A survey of the literature reveals only one known example of a nontrivial strongly clean element that is not uniquely strongly clean: namely, $-1$ in a unital ring in which 2 is a unit. In this note, we prove that no such examples exist in $\mathbb{M}_2(\mathbb{Z})$. In contrast, we show that examples of nontrivial strongly clean matrices that are not uniquely strongly clean, occur abundantly in matrix rings $\mathbb{M}_n(R)$ for $n \geq 3$.

## 1. Introduction

Throughout this paper, all rings are assumed to be associative with identity $1 \neq 0$. By $U(R)$ we denote the set of all the units of the ring $R$.

An element of a ring is called *clean* if it can be written as the sum of an idempotent and a unit. It is called *strongly clean* if, in such a decomposition, the idempotent and the unit commute. If this decomposition is unique (for example, if the idempotent is uniquely determined), the element is said to be *uniquely clean*.

Similarly, an element of a ring is called *nil-clean* if it is the sum of an idempotent and a nilpotent element. It is *strongly nil-clean* if the idempotent and the nilpotent commute, and *uniquely nil-clean* if this decomposition is unique (for instance, if the idempotent is unique). It is relatively easy to prove (see [3], Corollary 3.8) that if an element of a ring is strongly nil-clean, then it has exactly one strongly nil-clean decomposition.

The analogous statement for strongly clean elements fails. The only example commonly found in the literature (see [4] and, more recently, [5]) is the following.

In any ring with identity, for every idempotent $e^2 = e \in R$, the element $-e$ is strongly clean element, since

$$-e = (1 - e) + (-1).$$

If $2 \in U(R)$ (e.g., $R$ is a division ring), $-1$ has two distinct strongly clean decompositions, namely $-1 = 0 + (-1)$ and $-1 = 1 + (-2)$.

More generally, for any nonzero ring $R$, whenever a unit $u \in U(R) \cap (1 + U(R)$, there exists a unit $v$ such that $0 + u = 1 + v$, yielding two distinct strongly clean decompositions for the unit $u$. Such units are trivially clean (i.e., the idempotent component is either 0 or 1). They were recently termed *exceptional* units and studied in full generality in [1].

Since there exist clean elements with more than one strongly clean decomposition, following Chen et altri in [2], we say that an element $a \in R$ is *uniquely strongly clean* (USC) if it admits a unique strongly clean decomposition in $R$.

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Clearly, the strongly clean units mentioned above are not USC. On the other hand, as a general positive example, it was proved in [2] that idempotents in any ring are USC.

The main goal of this note is to prove that, apart from these exceptional units—which are precisely the trivial strongly clean elements—all the other strongly clean $2 \times 2$ integer matrices (that is, the nontrivial ones) are USC.

Somehow related to our discussion, it is worth mentioning from [2] (see Corollary 7): an $n \times n$ matrix ring is never USC for any $n > 1$.

Observe that this does not contradict our main result. Indeed, we show only that nontrivial strongly clean $2 \times 2$ integer matrices are USC; we do not claim that every $2 \times 2$ integer matrix admits a USC decomposition.

## 2. Proof of main result

We recall several useful facts. Also recall that a ring is called *GCD* if the greatest common divisor of every pair of elements exists.

**Lemma 2.1.** *(i) The properties of being clean, strongly clean, and uniquely strongly clean are invariant under conjugation. Hence, for matrices these are preserved under similarity.*

*(ii) Over a GCD domain, every nontrivial $2 \times 2$ idempotent matrix is similar to $E_{11}$. In particular, every nontrivial idempotent in $M_2(\mathbb{Z})$ is similar to $E_{11}$.*

*(iii) A $2 \times 2$ matrix commutes with $E_{11}$ if and only if it is diagonal.*

*Proof.* (i) Straightforward.

(ii) It can be checked that for $a \neq 0$,

$$P = \begin{bmatrix} \gcd(a,c) & \dfrac{b}{\gcd(a,b)} \\ -\dfrac{c}{\gcd(a,c)} & \gcd(a,b) \end{bmatrix}$$

and $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $bc = a(1-a)$, we get $\det(P) = 1$ and $PE = E_{11}P$. If $a = 0$ then $b = 0$ (with $E = \begin{bmatrix} 0 & 0 \\ c & 1 \end{bmatrix}$) or $c = 0$ (with $E = \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix}$), then by direct computation, $P = \begin{bmatrix} c & 1 \\ -1 & 0 \end{bmatrix}$ or $P = \begin{bmatrix} 0 & 1 \\ -1 & b \end{bmatrix}$ are suitable units, respectively.

(iii) Easy computation. □

Consequently, in any strongly clean decomposition involving $E_{11}$, the corresponding unit must be diagonal.

For convenience, we call a clean $2 \times 2$ matrix *basic* if it admits a decomposition $A = E+U$, with idempotent $E$ and diagonal unit $U$. Note that for integral matrices

$$U \in \{\pm I_2,\ \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}.$$

**Proposition 2.2.** *Every nontrivial strongly clean matrix in $M_2(\mathbb{Z})$ is similar to a basic diagonal matrix.*

*Proof.* Suppose $A$ is nontrivial strongly clean in $\mathbb{M}_2(\mathbb{Z})$. Then $A$ is similar to a strongly clean matrix $B$ whose idempotent is $E_{11}$, that is, $B = E_{11} + U$ with a unit $U = \begin{bmatrix} s & t \\ v & w \end{bmatrix}$, such that $E_{11}$ and $U$ commute. However the equalities $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} s & t \\ v & w \end{bmatrix} = \begin{bmatrix} s & t \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & t \\ v & w \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & 0 \\ v & 0 \end{bmatrix}$ hold if and only if $t = v = 0$, that is, if and only if $U$ is diagonal. Hence $s, w \in \{\pm 1\}$ and $U \in \{\pm I_2, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$ and so $B$ is basic. □

**Proposition 2.3.** *An integral basic matrix $E + U$ is strongly clean if and only if $E$ is diagonal.*

*Proof.* As previously mentioned, $U \in \{\pm I_2,\ \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}$, and since every $2 \times 2$ matrix commutes with $\pm I_2$, it remains to check the commutation with $\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. For instance, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ holds if and only if $b = c = 0$. Hence $E$ must be diagonal. □

With these preliminary facts, we are now ready to prove our main result.

**Theorem 2.4.** *All nontrivial strongly clean matrices in $\mathbb{M}_2(\mathbb{Z})$ are USC.*

*Proof.* As follows from the previous results, up to similarity, strongly clean matrices are sums of

$$E \in \{0_2, E_{11}, E_{22}, I_2\}$$

and

$$U \in \{\pm I_2,\ \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\}.$$

Thus there are 16 possible matrices. Suppose a matrix admits two strongly clean decompositions

$$E_1 + U_1 = E_2 + U_2.$$

Then

$$U_1 - U_2 = E_2 - E_1.$$

Since there are only four possible units, the possible differences $U_1 - U_2$ are

$$0_2,\ \pm 2I_2,\ \pm 2E_{11},\ 2E_{22},\ \pm 2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A direct verification shows that none of these nonzero matrices equals a difference of idempotents from $\{0_2, E_{11}, E_{22}, I_2\}$. Hence the equality above can occur only when both sides are zero, which implies $E_1 = E_2$ and $U_1 = U_2$.Therefore every nontrivial strongly clean matrix in $M_2(\mathbb{Z})$ is uniquely strongly clean. □

Observe that these strongly clean matrices are USC, but far from being uniquely clean.

**Example**. The matrix $A = E_{11} + I_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is uniquely strongly clean but of *infinite clean index*: $A = \begin{bmatrix} 1 & n \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$ is a clean decomposition, for any $n \in \mathbb{Z}$.

2.1. **Nontrivial strongly clean matrices that are not USC.** To obtain examples of nontrivial strongly clean matrices that are not USC over a unital ring, we should allow nontrivial idempotents in the base ring.

Take $R = \mathbb{Z}_{12}$ for which 4 and 9 are nontrivial idempotents. In $\mathbb{M}_2(\mathbb{Z}_{12})$, the scalar matrices $4I_2$ and $9I_2$ are central idempotents so, added with any units provide strongly clean $2 \times 2$ matrices.

To find some examples, we are searching for two units $U$ and $V$ such that $4I_2 + U = 9I_2 + V$. Equivalently, $U = 5I_2 + V$ and denote $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\det(V) = ad - bc \in U(\mathbb{Z}_{12})$. Then $\det(U) = \det \begin{bmatrix} 5+a & b \\ c & 5+d \end{bmatrix} = 1 + \det(V) + 5(a+d)$.

For a concrete example, assume $\det(V) = 1$. Hence $\det(U) = 1$ requires $1+5(a+d) = 0$. Equivalently, $a + d = 7$ and (say) $a = 3$, $d = 4$. Taking $b = c = 5 = 5^{-1}$, we get the desired example:

$$4I_2 + \begin{bmatrix} 8 & 5 \\ 5 & 9 \end{bmatrix} = 9I_2 + \begin{bmatrix} 3 & 5 \\ 5 & 4 \end{bmatrix}.$$

## 3. The exceptional units

Having established that all nontrivial strongly clean integral $2 \times 2$ matrices are USC, we recall for completeness the characterization of the exceptional units from [1] (Proposition 3.1, Corollary 3.2, and Proposition 3.9), namely the $2 \times 2$ invertible matrices that are strongly clean but not USC.

**Proposition 3.1.** *Let $R$ be a commutative ring, $U \in GL_n(R)$ and let $p_U(X) = \det(XI_n - U)$ be the characteristic polynomial of $U$. Then $U$ is an exceptional unit iff $p_U(1) \in U(R)$.*

**Corollary 3.2.** *A $2 \times 2$ matrix $U$ over a commutative ring $R$ is an exceptional unit iff $\det(U) \in U(R)$ and $1 - Tr(U) + \det(U) \in U(R)$.*

**Corollary 3.3.** *A $2 \times 2$ integral matrix $U$ is an exceptional unit iff $\det(U) = 1$ and $Tr(U) \in \{1, 3\}$, or else $\det(U) = -1$ and $Tr(U) \in \{-1, 1\}$.*

In closing, we provide some (general) trivial strongly clean matrices – specifically exceptional units over arbitrary rings – that are not USC. Here $r \in R$ is arbitrary.

**Examples**. 1) $\det(U) = 1 = Tr(U)$:

$$0_2 + \begin{bmatrix} r & 1 \\ r(1-r)-1 & -r+1 \end{bmatrix} = I_2 + \begin{bmatrix} r-1 & 1 \\ r(1-r)-1 & -r \end{bmatrix}.$$

2) $\det(U) = 1$, $Tr(U) = 3$:

$$0_2 + \begin{bmatrix} -r+1 & -1 \\ r(r+1)-1 & r+2 \end{bmatrix} = I_2 + \begin{bmatrix} -r & -1 \\ r(r+1)-1 & r+1 \end{bmatrix}.$$

3) $\det(U) = -1 = Tr(U)$:

$$0_2 + \begin{bmatrix} r & 1 \\ -r(r+1)+1 & -r-1 \end{bmatrix} = I_2 + \begin{bmatrix} r-1 & 1 \\ -r(r+1)+1 & -r-2 \end{bmatrix}.$$

4) $\det(U) = -1$, $Tr(U) = 1$:

$$0_2 + \begin{bmatrix} r & 1 \\ r(1-r)+1 & -r+1 \end{bmatrix} = I_2 + \begin{bmatrix} r-1 & 1 \\ r(1-r)+1 & -r \end{bmatrix}.$$

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