STRONGLY REGULAR ELEMENTS IN REDUCED RINGS ARE UNIT-REGULAR, AN UNDERGRADUATE PROOF

Definitions. An element a of a ring R is *strongly regular* if there is $x \in R$ such that $a^2x = a$ and is *unit-regular* if there a unit u such that $a = aua$.

A ring is *reduced* if it has no nonzero nilpotents, *Abelian* if its idempotents are central, and is *Dedekind finite* if one-sided invertible elements are two-sided (i.e., for all $a, b \in R : ab = 1 \Rightarrow ba = 1$).

For an idempotent e, $\overline{e} = 1 - e$ denotes the complementary idempotent.

Lemma 1. *(i) Reduced rings are Abelian.*
(ii) Abelian rings are Dedekind-finite.

Proof. (i) Let $e^2 = e \in R$ and $x \in R$. Computation shows that $(ex - exe)^2 = (xe - exe)^2 = 0$. Hence $ex = exe = xe$, i.e., $e \in Z(R)$.

(ii) Suppose $ab = 1$. Then $(ba)^2 = ba$ is an idempotent, so central by hypothesis. Thus $b = (ba)b = b(ba) = b^2a$ and so $1 = ab = ab^2a$.

Finally, $ba = (ab^2a)ba = (ab^2)(ab)a = ab^2a = 1$. □

The result in the title is well-known from more than 70 years (see [1]). In the sequel, we provide a undergraduate proof.

Theorem 2. *In any reduced ring, strongly regular elements are unit-regular.*

Proof. Suppose $a = a^2x$. Then $(a - axa)^2 = a^2 + axa^2xa - a^2xa - axa^2 = a^2 + axa^2 - a^2 - axa^2 = 0$, so $a = axa$, since R is reduced. Since reduced rings are Abelian, $e := ax$ is a central idempotent.

Consider $u = ex + \overline{e}$, $v = ea + \overline{e}$. Then (since $ea = ae$) $vu = eaex + \overline{e}^2 = e^2ax + \overline{e} = e + \overline{e} = 1$. Since reduced rings are also Dedekind finite, $uv = 1$ and so $u = v^{-1} \in U(R)$. Finally, from $a = a^2x = ae$ we get $a\overline{e} = 0$, so now $aua = a(ex + \overline{e})a = axa = a$. □

The existing proofs (see [1] or [2]) use the same steps, but use a graduate machinery in order to show that $ax = xa$, which is used (together with $e \in Z(R)$) in order to check $uv = 1$ ($uv = (ex+\overline{e})(ea+\overline{e}) = e(xea)+\overline{e}^2 = e(xa)+\overline{e} \overset{!}{=} e(ax)+\overline{e} = e+\overline{e} = 1$). Namely,

Arens, Kaplansky. If R is strongly regular, we deduce that R is semi-simple, that any primitive ideal M in R is maximal, and that $R - M$ is a division ring. It follows that, for the x satisfying $a^2x = a$, ax must map into either 0 or 1 modulo a maximal ideal. Hence $ax = xa$].

Lam **Ex.12.6C.** Consider any representation of R as a subdirect product of division rings $(\phi_i) : R \to \prod D_i$. Since $\phi_i(e) = \phi_i(a)\phi_i(x)$ is either 0 or 1 in D_i, we see that also $\phi_i(e) = \phi_i(x)\phi_i(a)$ (for every i), and hence $ax = xa$].

References

[1] R. F. Arens, I. Kaplansky *Topological representation of algebras*. Trans. Amer. Math. Soc. **63** (1948), 457-481.

[2] T. Y. Lam *Exercises in classical Ring Theory*. Problem Books in Math., Springer-Verlag New York Inc. 1995.