STRONGLY REVERSIBLE MATRICES

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1. Introduction

The rings we consider are associative with identity, $1 \neq 0$.

An *element* a of a ring R is called *strongly reversible* if $l_R(a) = r_R(a)$, that is, the left and the right annihilators coincide. Equivalently, if $ax = 0$ for some x, then $xa = 0$, *and conversely*. We can extend this definition as follows. Since the left annihilator is a left ideal and the right annihilator is a right ideal, it follows that this common annihilator actually is a (two-sided) ideal.

If the nilpotent elements of a ring are strongly reversible, we obtain the *nil-reversible* rings studied in [4]. It is proved that nil-reversible rings are Abelian (and 2-primal, weakly semicommutative and nil-Armendariz).

In this note, our goal is to describe the strongly reversible matrices, over several classes of rings.

2. General

Units are clearly strongly reversible, as $ax = 0$ implies $x = 0$.

Central elements are also clearly strongly reversible. The converse fails: in any noncommutative domain (e.g., the integer quaternions - Lipschitz), every nonzero element is strongly reversible.

Moreover, examples follow which show that there exist strongly reversible elements that are not units nor central (see diagonal matrices, next section).

Proposition 1. *The strongly reversible idempotents are central.*

Proof. One way is obvious. Conversely, recall that an idempotent is central in a ring R iff $eR\overline{e} = \overline{e}Re = 0$. Suppose e is not central. There exists $x \in R$ such that $\overline{e}xe \neq 0$. Then $e(\overline{e}xe) = 0$ but $(\overline{e}xe)e = \overline{e}xe \neq 0$, so e is not strongly reversible. The case $ex\overline{e} \neq 0$ is similar. □

Recall that for a, b elements in a ring R, b is (ring) *equivalent* to a if there exist units u, v such that $b = uav$.

Lemma 2. *(i) Strongly reversible elements are invariant under conjugation.*

(ii) If a is strongly reversible and $u \in U(R)$ then ua is strongly reversible.

(iii) If a is strongly reversible and $v \in U(R)$ then av is strongly reversible.

(iv) Strongly reversible elements are invariant under (ring) equivalence.

Proof. (i) Suppose a is strongly reversible and $uau^{-1}x = 0$ for a unit u and some $x \in R$. Then $au^{-1}x = 0$ and by hypothesis $u^{-1}xa = 0$ whence $xa = 0$. Again by hypothesis, $ax = 0$ and so $axu = 0$ and $xua = 0$. Finally, $xuau^{-1} = 0$, as desired.

(ii) Suppose $uax = 0$. Then (by left multiplication with u^{-1}), $ax = 0$. Hence $axu = 0$ and $xua = 0$ by hypothesis.

(iii) By (ii), va is strongly reversible whence, by (i), $av = v^{-1}(va)v$ is strongly reversible.

(iv) Follows from (ii) and (iii). □

2.1. **General, matrices.** As already mentioned, the annihilator of a strongly reversible matrix, is a two-sided ideal. As such, it is a matrix (sub)ring over an ideal of the base ring.

Proposition 3. *Over any simple ring, a matrix is strongly reversible iff it is zero or invertible.*

Proof. One way is clear, as the zero matrix and the invertible matrices are strongly reversible. Conversely, as a simple ring R has only the 0 and R ideals, any strongly reversible matrix must be invertible or zero, respectively. □

A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as $n \times n$ matrix. When $n = 1$, $S = 0$.

Lemma 4. *(i) Let A be an $n \times n$ matrix over any ring R. If A is strongly reversible, so is the transpose A^T.*

(ii) The shift matrices are not strongly reversible.

(iii) A (finite) direct sum of matrices is strongly reversible may not be strongly reversible

Proof. (i) Assume $A^T X = 0$ for some $X \in \mathbb{M}_n(R)$. Then the transpose $X^T A = 0$ and by hypothesis $AX^T = 0$. By transpose again, $XA^T = 0$.

(ii) As $S = E_{12} + E_{23} + ... + E_{n-1,n}$, clearly $SE_{11} = 0$ but $E_{11}S = E_{12} \neq 0$.

(iii) Over $\mathbb{Z}_4$ take the direct sum of the 1×1 unit matrix 1 and the central 2×2 matrix $2I_2$. As already mentioned, any unit and any central element are strongly reversible. However, the direct sum $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not strongly reversible in $\mathbb{M}_3(\mathbb{Z}_4)$. Indeed, $M \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} = 0_3 \neq \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} M$. □

2.2. **Idempotent matrices.** Recall that a ring is called *connected* if it has only the trivial idempotents. Proposition 1 has an immediate consequence.

Proposition 5. *Over any ring R, the only strongly reversible $n \times n$ matrices are eI_n (i.e., the scalar matrices), for some central idempotent $e \in R$.*

Corollary 6. *Over any connected ring, the only strongly reversible idempotent $n \times n$ matrices are the trivial ones.*

Corollary 7. *Over any domain (in particular, over $\mathbb{Z}$), the only strongly reversible idempotent matrices are the trivial ones.*

2.3. **Nilpotent matrices.** The above mentioned nonequality $E_{11}E_{21} = 0 \neq E_{21} = E_{21}E_{11}$ also shows that the nilpotent 2×2 matrix E_{21} is not strongly reversible. Similarly, E_{12} is not strongly reversible.

This can be twice further generalized, as follows.

Proposition 8. *Over any GCD commutative domain, the only strongly reversible nilpotent 2×2 matrix is the zero matrix.*

Proof. Follows using Lemma 2, since every nonzero nilpotent is similar to a ring multiple of E_{12} (e.g., see [2], P. 4.3), which as seen above, is not strongly reversible. □

Recall from [2] (T. 3.3) the following characterization.

Theorem 9. *The following are equivalent for a ring R:*

(1) Every nilpotent matrix over R is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).

(2) R is a division ring.

Then we can prove the following result.

Theorem 10. *Over any division ring R, the only strongly reversible nilpotent $n \times n$ matrix is the zero matrix.*

Proof. According to Lemma 4 (ii) and the previous theorem, it suffices to show that any direct sum of shift matrices, possibly of different sizes, is not strongly reversible. To see this, we just extend the proof of Lemma 4 (ii). Namely, for a given direct sum σ of shift matrices, we consider the diagonal matrix D whose nonzero entries are the left upper corners of the blocks corresponding to the diagonal shift matrices, which are equal to 1. Then $\sigma D = 0 \neq D\sigma$.

To give an example, consider the following direct sum of shift matrices (of sizes 2, 1 and 3):

$$\sigma = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and the diagonal matrix

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\sigma D = 0_6$ but $D\sigma = E_{12} + E_{45}$. □

2.4. **Factor rings, products etc.** Strongly reversibility *behaves badly towards factor rings.* If $a \in R$ is strongly reversible and I is an ideal of R, $a + I$ may not be strongly reversible in R/I. The converse also generally fails.

[We cannot relate $ax = 0$ implies $xa = 0$, with $ax \in I$ implies $xa \in I$]. Examples can be reproduced from [4]

A homomorphic image of a nil-reversible ring may not be nil-reversible. In particular, the image of a strongly reversible element may not be strongly reversible.

Example 2.17. Let $R = D[x, y]$, where D is a division ring and $J =< xy >$, where $xy \neq yx$. As R is a domain, R is reversible. In particular, yx is strongly reversible. Clearly $\overline{yx} \in N(R/J)$ and $\overline{x}(\overline{yx}) = \overline{xyx} = 0$. But, $(\overline{yx})\overline{x} = \overline{yx^2} \neq 0$. This implies R/J is not nil-reversible. In particular, $\overline{yx}$ is not strongly reversible.

Proposition 11. *Let R_i be rings for every $i \in I$. An element $(a_i) \in \prod_{i\in I} R_i$ is strongly reversible iff a_i is strongly reversible in R_i, for every $i \in I$.*

Proof. Obvious. □

3. 2 × 2 matrices

3.1. **Diagonal.** It is easy to characterize mostly all strongly reversible *diagonal* 2×2 matrices (in particular, all the strongly reversible diagonal 2×2 matrices over commutative rings).

Proposition 12. *Let a, b be central elements in a ring R. The diagonal matrix $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ is strongly reversible iff the annihilators $r(a) = r(b)$. In particular, D is strongly reversible if a and b are associated (in divisibility).*

Proof. Assume $DX = 0_2$ for some $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Equivalently, $ax = ay = bz = bt = 0$. Next, $XD = 0_2$ is equivalent to $ax = az = by = bt = 0$. Hence D is strongly reversible iff $ay = 0$ implies $by = 0$ and $bz = 0$ implies $az = 0$. Equivalently, the (say, right) annihilators $r(a) = r(b)$. Central elements associated (in divisibility) have equal annihilators. □

Corollary 13. *Let R be a ring, $a \in R\backslash U(R)$ a central element and $1 \neq u \in U(R)$. Then $D = \begin{bmatrix} a & 0 \\ 0 & ua \end{bmatrix}$ is strongly reversible, not invertible nor central.*

Proof. Clearly $r(a) = r(ua)$, so D is strongly reversible by the previous proposition. Since $q \neq u$, D is not central. Finally D is not invertible because a is not a unit. □

Exercise 14. *Let R be a ring such that char$(R) \neq 4$ and 4 not a unit in R. Then $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ is strongly reversible over R, not invertible, nor central.*

Proof. According to Proposition 12, A is strongly reversible.

As 4 is not a unit, A is not invertible. Since $4 \neq 0$, it is not central: $A(E_{11} + E_{12}) = 2(E_{11} + E_{12}) \neq 2(E_{11} - E_{12}) = (E_{11} + E_{12})A$. □

In particular, $\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ is strongly reversible over $\mathbb{Z}_6$, not invertible, nor central.

In general, two elements of a ring may have the same annihilator without being associated. Equality of annihilators does not imply equality of the principal ideals they generate.

However, there are classes of rings where this converse holds. Namely,

...

The previous proposition generalizes to the $n \times n$ case. We call *pairwise associated* (in divisibility) a set of elements S if every two elements $a, b \in S$ are associated.

Proposition 15. *Let $a_1, ..., a_n$ be central elements in a ring R. The diagonal matrix $D = diag(a_1, ...a_n)$ is strongly reversible iff the annihilators $r(a_1) = ... = r(a_n)$. In particular, D is strongly reversible if $a_1 ..., a_n$ are pairwise associated (in divisibility).*

3.2. **Triangular.** We continue with the upper triangular 2×2 strongly reversible matrices.

Proposition 16. *Let a, b, d be central elements in a ring R. The upper triangular matrix $T = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is strongly reversible iff $r(a) = r(b) = r(c)$.*

Proof. Denote $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ and $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$. Then, $AX = 0_2$ is equivalent to the linear system $\begin{cases} ax + bz &= 0 \\ ay + bt &= 0 \\ dz &= 0 \\ dt &= 0 \end{cases}$ and $XA = 0_2$ is equivalent to the linear system $\begin{cases} ax &= 0 \\ bx + dy &= 0 \\ az &= 0 \\ bz + dt &= 0 \end{cases}$.

For $ax + bz = 0$ to imply $ax = 0$ (or $dt = 0$ to imply $bz + dt = 0$) it is necessary $bz = 0$. Moreover $r(a) \subseteq r(d)$ and $r(a) \subseteq r(b)$ are necessary and since for strongly reversiblity of A $AX = 0_2$ is equivalent to $XA = 0_2$, these should be equalities.

It remains to relate to these equalities (i.e., $ax = az = bz = dz = dt = 0$), $ay + bt = 0$ implies $bx + dy = 0$.

From $ax = 0$ we get $x \in r(a) = r(b)$ and so $bx = 0$. Finally, from $dt = 0$ we obtain $t \in r(d) = r(b)$ and so $bt = 0$. The only remaining implication is $dy = 0$ whenever $ay = 0$. But this holds as $r(a) = r(d)$. □

Remark. Let $A \in \mathbb{M}_n(R)$ be strongly reversible. Then there exists an ideal I of R such that $Ann_{\mathbb{M}_n(R)}(A) = \mathbb{M}_n(I)$. Clearly, $\mathbb{M}_n(I)$ contains all the scalar matrices aI_n and so $Aa = aA = 0$ for every $a \in I$.

Conversely, if there is an ideal I of R, *maximal* with the property that $Aa = aA = 0$ for every $a \in I$, then A is strongly reversible.

References

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