Any strongly regular ring is regular: an elementary proof

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A well-known proof for the claim goes like this:

1) any strongly regular ring is reduced, and so semiprime;

2) for any prime ideal P of a ring, R/P is a division ring;

3) in any division ring $\alpha^2\beta = \alpha$ implies $\alpha\beta\alpha = \alpha$;

4) The proof: suppose $a = a^2b$ for elements $a, b \in R$. For any prime ideal P, we have $\overline{a} = \overline{a}^2\overline{b}$ in R/P so according to (3), $\overline{a} = \overline{a}\overline{b}\overline{a}$. Hence $aba - a \in P$ for every prime ideal of R. As R is semiprime, the intersection of all the prime ideals is 0 and so $aba - a = 0$, as desired.

In the sequel we provide **an elementary proof** which avoids semiprime rings and prime ideals.

P1. A strongly regular ring is reduced.

Proof. Obviously $a = a^2x$ implies $a^3x^2 = a^2x = a$ and $a^{n+1}x^n = a$ where n is an arbitrary positive integer. Therefore $a^n = 0$ implies $a = 0$, for any element a.

P2. Any strongly regular ring is regular.

Proof. Let a be an arbitrary element of R. Then $a = a^2x$ implies $(a - axa)^2 = 0$. Hence by P1, $a = axa$.