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ON SQUARE STABLE RANGE ONE MATRICES OVER COMMUTATIVE RINGS

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Abstract. In [7], Khurana and Lam introduced the concept of left square stable range one (ssr1) for an element of a unital ring. In this paper, over commutative rings, we examine 2×2 matrices that satisfy the ssr1 condition. Our findings indicate significant differences from the stable range one (sr1) condition, necessitating the development of specialized techniques.

Among our results, we provide characterizations of 2×2 matrices that possess ssr1 in several cases: implicitly over commutative rings, nilpotent matrices over commutative reduced rings, and explicitly over elementary divisor domains.

As applications, we demonstrate that over commutative Bézout domains, ring multiples of idempotent 2×2 matrices have ssr1. Additionally, we characterize ssr1 matrices with a zero row (or zero column) and offer an explicit description of ssr1 integral matrices.

Building on these results, we further show that the Jacobson Lemma for ssr1 holds for 2×2 integral matrices, contingent on a conjecture regarding the greatest common divisors of their entries.

1. Introduction

In [7], a variation of the stable range one condition (sr1) for elements and rings was introduced as follows: an element a of a unital ring R has *left square stable range one* (ssr1) if, for each $b \in R$ such that $Ra + Rb = R$ there exists $y \in R$ such that $a^2 + yb$ is a unit of R. A ring R is said to have left ssr1 if all its elements

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satisfy this property. The condition for right ssr1 for elements and rings is defined symmetrically.

Recently, in [6], it was shown that this condition is left-right symmetric for elements (and, by extension, for rings) by applying the so-called Super Jacobson's Lemma. Consequently, throughout the rest of this paper, we will omit the attribute "left" when discussing ssr1 elements. We denote by $ssr1(R)$ the ssr1 elements of a ring R and by $sr1(R)$ the sr1 elements of a ring R.

As usual, for any positive integer $n \geq 2$, E_{ij} (with $1 \leq i, j \leq n$) denotes the $n \times n$ matrix with all entries zero except for the (i, j)-entry, which equals 1. A simple example using 2×2 matrices demonstrates that there are significant differences between sr1 and ssr1.

For any unital ring, we have $E_{12} = E_{11}(E_{12} + E_{21})$, indicating that a nilpotent element can be expressed as the product of an idempotent and a unit. From the previous definition, it follows that both idempotents (as these have sr1; see also Lemma 2.1) and units have ssr1. However, we will show (see Example 2.7, Section 2) that E_{12} does not have ssr1. Therefore:

(i) The statement "All matrices rE_{ij} have sr1 over any (unital) ring" (e.g., see [1] or [7]) does not hold for ssr1 elements. In particular, there exist 2×2 matrices with three zero entries that do not have ssr1.

(ii) Products of ssr1 elements may not have ssr1.

(iii) Generally, $sr1(R) \subsetneqq ssr1(R)$.

(iv) Unit-regular elements (which are known to be products of an idempotent and a unit, and have sr1) may not have ssr1.

(v) The ssr1 condition is not invariant under equivalences.

Two elements a, b of a ring R are said to be *equivalent* if there exist units $p, q \in R$ such that $b = paq$. Notably, points (ii) and (v) above have important implications when comparing ssr1 with sr1.

Since diagonal reduction for matrices involves equivalences, it cannot be applied when studying ssr1 for matrices. The same obstacle arises when attempting to prove results over elementary divisor rings, particularly over $\mathbb{Z}$. When reducing diagonal matrices to scalar (diagonal) matrices, we use the fact that products of sr1 matrices must also have sr1. Thus, this is also not applicable when studying ssr1. Therefore, the study of ssr1 for matrices - the main focus of this paper - requires different methods to achieve analogous (or non-analogous) results.

In fact, as already mentioned in [7], the properties sr1 and ssr1 are logically independent for general rings.

One might consider diagonalization of matrices using eigenvalues and eigenvectors in these conditions. However, as is well-known, this approach is highly specialized and would overly restrict our study, even for integral 2×2 matrices.

The structure of this paper is as follows: In Section 2, we present the main results concerning ssr1 matrices. We begin by characterizing ssr1 for 2×2 matrices over commutative rings, with a particular focus on matrices with zero determinant. Additionally, we demonstrate that for commutative reduced rings, a nilpotent 2×2 matrix has ssr1 if and only if its entries lie within the Jacobson radical. The section concludes with an explicit characterization of ssr1 2×2 matrices over EDDs (elementary divisor domains).

Section 3 explores several applications of the results established in Section 2. In particular, we prove that over commutative Bézout domains, ring multiples of nontrivial idempotent 2×2 matrices have ssr1. We also provide a characterization of 2×2 matrices with either a zero row or zero column that possess ssr1. Furthermore, we describe integral 2×2 matrices that exhibit ssr1 and address the Jacobson Lemma for ssr1 matrices (i.e., if $ssr(1 - ab) = 1$, then $ssr(1 - ba) = 1$). The Jacobson Lemma holds for $\mathbb{M}_2(\mathbb{Z})$ under the assumption that a conjecture regarding the greatest common divisor (gcd) of the entries is true.

In Section 4, we present explicit square unitizers (the element y in the first definition provided above) for the integral matrices $\begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Throughout this paper, all base rings are assumed to be associative, commutative, and possess an identity element. To simplify the writing, in all our results, the commutativity hypothesis of the base rings will not be mentioned.

Many results also require the rings to be domains, specifically Bézout domains or EDDs. The set of all units in a ring R is denoted by $U(R)$, while $J(R)$ represents the Jacobson radical of R. For simplicity, we refer to two elements a and b in a ring as *coprime* if their gcd exists and equals 1. Whenever appropriate, we use the abbreviation "iff" for "if and only if." The terms GCD (greatest common divisor) rings (or domains) and EDD (elementary divisor domains) are also used throughout the paper. Note that the (commutative) EDDs are Bézout domains and the Bézout domains are GCD domains.

2. The main results

An element a of a ring R is said to have left *square stable range 1* (abbreviated as *ssr1*) if for each $b \in R$ such that $Ra + Rb = R$, there exists $y \in R$ such that $a^2 + yb \in U(R)$. In this case, we write $ssr(a) = 1$. Equivalently, $ssr(a) = 1$ iff for every $x \in R$, there exists $y \in R$ such that $a^2 + y(1 - xa) \in U(R)$.

As previously mentioned, we will omit the term "left" when referring to ssr1 elements. Following the notation in [1], for simplicity, we refer to y as a (*square*) *unitizer* for a depending on x.

It is important to note that matrix rings $\mathbb{M}_n(R)$ for $n > 1$ do not have ssr1 (see [7], Theorem 4.1 (4)). Hence, when dealing with matrices, we will only consider ssr1 elementwise.

We begin with a useful result.

Lemma 2.1. *(i) Strongly regular elements have ssr1. In particular, idempotents and units of any ring have ssr1.*

(ii) The ssr1 elements are invariant to conjugations.

(iii) If a has ssr1, so is $-a$.

(iv) The ssr1 elements are not invariant to equivalences.

However, if a has ssr1 and commutes with $u \in U(R)$, then both au and ua have ssr1.

Proof. (i) A proof is given in [7] (Theorem 5.2). The special cases are obvious ($e^2 = e$ resp. $y = 0$), taking into account that idempotents have sr1.

(ii) For every x there is a y such that $a^2 + y(1 - xa) \in U(R)$. It follows that $u^{-1}[a^2 + y(1 - xa)]u \in U(R)$ whence $(u^{-1}au)^2 + u^{-1}yu[1 - (u^{-1}xu)(u^{-1}au)] \in U(R)$.

(iii) Suppose $ssr(a) = 1$, that is, for every x there is y with $a^2 + y(1 - xa) \in U(R)$. Then for every $-x$ there is a z such that $(-a)^2 + z(1 + xa) \in U(R)$, so also $ssr(-a) = 1$ holds.

(iv) The negative claim follows from Example 2.7. As for the positive one, assume $ssr(a) = 1$ and $au = ua$ for some $u \in U(R)$. Start with $x(ua) + b = 1$. Since a has ssr1, there is $y \in R$ such that $a^2 + yb \in U(R)$. By left multiplication with u^2 we get $u^2a^2 + (u^2y)b = (ua)^2 + (u^2y)b \in U(R)$, so $ssr(ua) = 1$. Finally, for $ssr(au) = 1$ we use the left-right symmetry of ssr1 and right multiplication with u^2. □

It is worth noting that while the transpose of an invertible matrix may not always be invertible, it is invertible over commutative rings. Furthermore, over any commutative ring, a matrix A has (left) ssr1 iff the transpose A^T has (right) ssr1.

In our first main result, we characterize the 2×2 square stable range 1 matrices over any (commutative) ring. Since the next result involves significant computation, we opt to use $XA - I_2$ instead of $I_2 - XA$, as this choice reduces the number of negative signs (equivalent to changing the sign of Y).

Theorem 2.2. *Let R be a ring and $A \in \mathbb{M}_2(R)$. Then A has square stable range one iff for any $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that*

$$\det(Y)[\det(X)\det(A) - \mathrm{Tr}(XA) + 1] + \det(A)[\mathrm{Tr}(X\mathrm{adj}(A)Y) + \det(A)] - \mathrm{Tr}(A^2\mathrm{adj}(Y))$$

is a unit of R. Here $\mathrm{adj}(Y)$ is the classical adjoint (also called the adjugate) of Y.

Proof. As mentioned, A has ssr1 in $\mathbb{M}_2(R)$ iff for every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$ there is $Y = \begin{bmatrix} x & y \\ z & t \end{bmatrix} \in \mathbb{M}_2(R)$ such that $A^2 + Y(XA - I_2)$ is invertible. Since the base ring is supposed to be commutative, $A^2 + Y(XA - I_2)$ is invertible in $\mathbb{M}_2(R)$ iff $\det(A^2 + Y(XA - I_2))$ is a unit of R. For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the computation amounts to the determinant of the 2×2 matrix with columns

$$C_1 = \begin{bmatrix} a_{11}^2 + a_{12}a_{21} + (aa_{11} + ba_{21} - 1)x + (ca_{11} + da_{21})y \\ a_{21}(a_{11} + a_{22}) + (aa_{11} + ba_{21} - 1)z + (ca_{11} + da_{21})t \end{bmatrix}$$

and

$$C_2 = \begin{bmatrix} a_{12}(a_{11} + a_{22}) + (aa_{12} + ba_{22})x + (ca_{12} + da_{22} - 1)y \\ a_{12}a_{21} + a_{22}^2 + (aa_{12} + ba_{22})z + (ca_{12} + da_{22} - 1)t \end{bmatrix}.$$

In computing this determinant, there are several terms we gather as follows:

the coefficient of xz: $(aa_{11}+ba_{21}-1)(aa_{12}+ba_{22})-(aa_{11}+ba_{21}-1)(aa_{12}+ba_{22})$, which equals zero,

the coefficient of xt: $(aa_{11} + ba_{21} - 1)(ca_{12} + da_{22} - 1) - (ca_{11} + da_{21})(aa_{12} + ba_{22}) = \det(X)\det(A) - aa_{11} - ba_{21} - ca_{12} - da_{22} + 1$

the coefficient of yz: $(ca_{11}+da_{21})(aa_{12}+ba_{22})-(aa_{11}+ba_{21}-1)(ca_{12}+da_{22}-1) = -\det(X)\det(A) + aa_{11} + ba_{21} + ca_{12} + da_{22} - 1$

the coefficient of yt: $(ca_{11}+da_{21})(ca_{12}+da_{22}-1)-(ca_{11}+da_{21})(ca_{12}+da_{22}-1)$, which equals zero,

and another five terms

$(a_{11}^2 + a_{12}a_{21})[(aa_{12} + ba_{22})z + (ca_{12} + da_{22} - 1)t]$,

$(a_{12}a_{21} + a_{22}^2)[(aa_{11} + ba_{21} - 1)x + (ca_{11} + da_{21})y]$,

$-a_{12}(a_{11} + a_{22})[(aa_{11} + ba_{21} - 1)z + (ca_{11} + da_{21})t]$,

$-a_{21}(a_{11} + a_{22})[(aa_{12} + ba_{22})x + (ca_{12} + da_{22} - 1)y]$,

$\det(A^2) = \det^2(A)$.

Then this determinant is

$$
\begin{gathered}
\det(Y)(\det(X)\det(A) - aa_{11} - ba_{21} - ca_{12} - da_{22} + 1)+ \\
+(a_{11}^2 + a_{12}a_{21})[(aa_{12} + ba_{22})z + (ca_{12} + da_{22} - 1)t]+ \\
+(a_{12}a_{21} + a_{22}^2)([(aa_{11} + ba_{21} - 1)x + (ca_{11} + da_{21})y]- \\
-a_{12}(a_{11} + a_{22})[(aa_{11} + ba_{21} - 1)z + (ca_{11} + da_{21})t]- \\
-a_{21}(a_{11} + a_{22})[(aa_{12} + ba_{22})x + (ca_{12} + da_{22} - 1)y] + \det^2(A)
\end{gathered}
$$

or

$$
\begin{gathered}
\det(Y)[\det(X)\det(A) - \mathrm{Tr}(XA) + 1]+ \\
+\det(A)[(aa_{22} - ba_{21})x + (ba_{11} - aa_{12})z + (ca_{22} - da_{21})y + (da_{11} - ca_{12})t + \det(A)] \\
-(a_{11}^2 + a_{12}a_{21})t + a_{12}(a_{11} + a_{22})z + a_{21}(a_{11} + a_{22})y - (a_{12}a_{21} + a_{22}^2)x.
\end{gathered}
$$

Finally this gives the condition in the statement. □

This theorem can be used to obtain the left-right symmetry of ssr1 in this particular case.

Corollary 2.3. *Let R be a ring and $A \in \mathbb{M}_2(R)$. Then A has left square stable range 1 iff A has right square stable range 1.*

Proof. Using the properties of determinants, the properties of the trace and the commutativity of the base ring, it is readily seen that changing A, X, Y into transposes and reversing the order of the products does not change the condition in the previous theorem. □

In the reminder of the paper, a special case of this characterization theorem will often be used.

Corollary 2.4. *Let R be a ring and $A \in \mathbb{M}_2(R)$ with $\det(A) = 0$. Then A has square stable range 1 iff for any $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that*

$$\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(A)\mathrm{Tr}(A\mathrm{adj}(Y)) \in U(R).$$

Proof. As $\det(A) = 0$, just notice that $A^2 = \mathrm{Tr}(A)A$, by Cayley-Hamilton's theorem. □

For simplicity, the equation displayed in the above corollary will be referred to as *the XY-equation of A*. In domains where equalities are considered modulo association, we replace membership in $U(R)$ with equality to 1.

As a first application of this corollary, we characterize the nilpotent 2×2 matrices that have ssr1 over (commutative) reduced rings,

Recall (see [4]) that an element s of a ring R is *quasi-nilpotent* if $1 - sa \in U(R)$ for every $a \in R$ that commutes with s. In a commutative ring, s is quasi-nilpotent iff $1 - Rs \subseteq U(R)$ iff $s \in J(R)$. In other words, the quasi-nilpotent elements are exactly the elements of the Jacobson radical of R.

Proposition 2.5. *Let $T \in \mathbb{M}_2(R)$ be nilpotent over a reduced ring R. The following statements are equivalent.*

(a) $ssr(T) = 1$;

(b) For every $X \in \mathbb{M}_2(R)$, $1 - \mathrm{Tr}(XT)$ is a unit;

(c) The entries of T belong to the Jacobson radical.

The reduced hypothesis may be dropped for nilpotent 2×2 matrices that have zero determinant.

PROOF. (a) $\Leftrightarrow$ (b). As T is nilpotent, $T^2 = 0_2$, and since the ring R is reduced, $\det(T) = 0$. Since for zero determinant matrices, ssr1 is characterized by Corollary 2.4, by replacement, $ssr(T) = 1$ iff for every $X \in \mathbb{M}_2(R)$ there exists $Y \in \mathbb{M}_2(R)$ such that $\det(Y)(1 - \mathrm{Tr}(XT)$ is a unit. As one can always choose $Y = I_2$, and the base ring is commutative, this reduces to $1 - \mathrm{Tr}(XT)$ is a unit.

(b) $\Leftrightarrow$ (c). One way, denoting $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $T = [t_{ij}]$, (b) amounts to $1 - at_{11} - bt_{21} - ct_{12} - dt_{22}$ being a unit. By choosing zero, three out of a, b, c, d, it follows that the entries of T must be quasi-nilpotents, and so, the base ring being commutative, must belong to the Jacobson radical of R.

Conversely, if all $t_{ij} \in J(R)$, so is $\mathrm{Tr}(XT) = at_{11} + bt_{21} + ct_{12} + dt_{22} \in J(R)$ for every a, b, c, d. Hence $1 - \mathrm{Tr}(XT)$ is a unit and so (b) and (c) are equivalent. $\square$

Remark. There exist nilpotent 2×2 matrices with a nonzero determinant (and consequently, a nonzero trace). To construct a diagonal example, we need two elements whose squares are zero, but whose product is nonzero. Some specific examples include:

a) The matrices E_{12} and E_{21} in $\mathbb{M}_2(R)$ for any ring R, or

b) The polynomials $2 + (X^2)$ and $X + (X^2)$ in $\mathbb{Z}_4(X)/(X^2)$.

Recall that a ring is called *semiprimitive* (or Jacobson semisimple) if the Jacobson radical is zero.

Corollary 2.6. *Over any semiprimitive ring, 0_2 is the only 2×2 nilpotent matrix which has ssr1. In particular, 0_2 is the only 2×2 nilpotent integral matrix which has ssr1.*

PROOF. Just note that commutative semiprimitive rings are reduced. $\square$

Example 2.7. *Let R be any ring and $s \notin J(R)$. Then $ssr_{\mathbb{M}_2(R)}(sE_{12}) \neq 1$. In particular, this holds if $s \in U(R)$, including $s = 1$.*

Since $(sE_{12})^2 = 0_2$, we use the previous proposition for the first part. In particular, if $s \in U(R)$, for $X = s^{-1}E_{21}$, $1 - \mathrm{Tr}(XT) = 0$ is not a unit.

Note that for semiprimitive rings (incl. $\mathbb{Z}$), the statement holds whenever $s \neq 0$. This example also shows that the 2×2 matrices with three zero entries (have sr1 but) may not have ssr1.

Remark 2.8. *If $ssr(ab) = 1$ then a or b may not have ssr1. This may happen even if one of a, b is a unit.*

Indeed, for the unit $U = E_{12}+E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, both $E_{12}U = E_{11}$, $UE_{12} = E_{22}$ are idempotents, so have ssr1.

To explicitly characterize the 2×2 matrices that have ssr1, we must address two key challenges.

First, we need to determine when the XY-equation is solvable. Second, given two elements u and v, along with a 2×2 matrix B (with zero determinant and entries that are coprime, if necessary), we need to find a 2×2 matrix V such that $\det(V) = u$ and $\mathrm{Tr}(BV) = v$. The adjugate of V provides the explicit square unitizer Y for B.

In this paper, we characterize the solvability of the XY-equation over Bézout domains in Theorem 2.10. Regarding the matrix V, it is explicitly constructed in Lemma 2.11 for Bézout domains when B is a nontrivial idempotent, and implicitly for matrices with zero determinant and coprime entries over EDDs in Theorem 2.15.

In the following section, we present several applications of these results, starting with multiples of nontrivial idempotents, where explicit square unitizers are found. Additionally, Section 5 provides explicit square unitizers for the integral matrix $\begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$ and sketches solutions for $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

The following result will be useful.

Lemma 2.9. *Let R be a Bézout domain and $1 \neq a \in R$. There exist an element x such that $\gcd(a, bx-1) \neq 1$ for some $b \in R$ iff there exists a prime divisor of a not dividing b.*

Proof. One way, let p be a prime divisor of $\gcd(a, bx-1)$. Then $p \mid a$ and $p \mid bx - 1$ and so $p \nmid b$. Conversely, assume there exists a prime p with $p \mid a$ and $p \nmid b$. Then $\gcd(p, b) = 1$ and so there exist elements y, x such that $yp + xb = 1$. It follows that $p \mid bx - 1$, as desired. $\square$

For a matrix A, denote by $\gcd(A)$ the greatest common divisor of the entries of A. If $\gcd(A) = 1$ we say that the entries of A are coprime. Alternatively, we use the term *unimodular row* for $(\alpha, \beta, \gamma, \delta)$ if $\gcd(\alpha, \beta, \gamma, \delta) = 1$.

We are now ready to prove the characterization of zero determinant 2×2 matrices whose XY-equation is solvable.

Theorem 2.10. *Let A be a nonzero 2×2 matrix over a Bézout domain, $\det(A) = 0$ and $\mathrm{Tr}(A) \neq 0$. For $\delta = \gcd(A)$, write $A = \delta B$. Then the XY-equation of A is solvable iff $\mathrm{Tr}(B) = 1$ (equivalently, if B is idempotent) or else, all prime divisors of $\mathrm{Tr}(B)$ divide δ.*

Proof. As $\det(A) = 0$, we get $\det(B) = 0$ and so by Cayley-Hamilton's theorem, $A^2 = \mathrm{Tr}(A)A$ and $B^2 = \mathrm{Tr}(B)B$. Hence $\mathrm{Tr}(A^2 adj(Y)) = \delta^2 \mathrm{Tr}(B)\mathrm{Tr}(B adj(Y))$.

By Corollary 2.4, the XY-equation is now

$$\det(Y)(\delta \mathrm{Tr}(XB) - 1) + \delta^2 \mathrm{Tr}(B)\mathrm{Tr}(B adj(Y)) = 1 \qquad (*).$$

Note that as $\gcd(B) = 1$, for every $r \in R$, there exists $X \in \mathbb{M}_2(R)$ such that $\mathrm{Tr}(XB) = r$.

Indeed, $\mathrm{Tr}(XB) = ab_{11} + bb_{21} + cb_{12} + db_{22} = r$ is a linear Diophantine equation with $\gcd(b_{11}, b_{21}, b_{12}, b_{22}) = 1$ (here $B = [b_{ij}]$). Hence it is solvable for a, b, c, d.

Also note that $\gcd(\delta \mathrm{Tr}(XB) - 1, \delta^2) = 1$ so that the equation (*) is *not* solvable iff there exists X such that $\gcd(\delta \mathrm{Tr}(XB) - 1, \mathrm{Tr}(B)) \neq 1$. Since (as mentioned above) $\mathrm{Tr}(XB)$ ranges over all the elements of R, the equation (*) is not solvable iff there exists an element $x \in R$ such that $\gcd(\delta x - 1, \mathrm{Tr}(B)) \neq 1$. As witnessed by Lemma 2.9, this holds iff $\mathrm{Tr}(B)$ has a prime divisor which divides not δ.

By denial, the XY-equation is solvable iff all prime divisors of $\mathrm{Tr}(B)$ divide δ. □

Remarks. 1) The previous theorem is primarily useful in the negative case. If a prime divisor of $\mathrm{Tr}(B)$(where $\mathrm{Tr}(B) \neq 1$) does not divide δ, then not only is the XY-equation unsolvable, but also $ssr(A) \neq 1$.

2) The case where $\mathrm{Tr}(B) = 1$ (i.e., B is idempotent) is addressed separately in Theorem 3.1 in the next section, as an application of the following lemma.

Next, we prove a technical but useful result.

Lemma 2.11. *Let E be a nontrivial idempotent 2×2 matrix over a GCD domain R and $u, v \in R$. There exists a 2×2 matrix V such that $\det(V) = u$ and $\mathrm{Tr}(EV) = v$.*

PROOF. Let $E = \begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}$ with $\alpha(1-\alpha) = \beta\gamma$ be a nontrivial idempotent matrix over a (commutative) GCD domain R and let $e : R^2 \to R^2$ be the corresponding projection (idempotent endomorphism). Thus

$$e(\begin{bmatrix} x \\ y \end{bmatrix}) = E\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \alpha x + \beta y \\ \gamma x + (1-\alpha)y \end{bmatrix}$$

and $R^2 = im(e) \oplus \ker(e) = im(e) \oplus im(1_{R^2} - e)$.

A basis $\{f_1, f_2\}$ for $im(e)$ is obtained by first solving $e(f_1') = f_1'$ and $e(f_2') = 0$, respectively. Solving the corresponding homogeneous linear systems gives $f_1' \in \{\begin{bmatrix} \alpha \\ \gamma \end{bmatrix}, \begin{bmatrix} \beta \\ 1-\alpha \end{bmatrix}\}$ and $f_2' \in \{\begin{bmatrix} -\beta \\ \alpha \end{bmatrix}, \begin{bmatrix} -1+\alpha \\ \gamma \end{bmatrix}\}$.

Case 1. Assume $\gamma \neq 0$. To obtain a basis (and the corresponding change-of-basis matrix P) we choose $f_1' = \begin{bmatrix} \alpha \\ \gamma \end{bmatrix}$, $f_2' = \begin{bmatrix} -1+\alpha \\ \gamma \end{bmatrix}$ and (to have unimodular columns in P) we divide the components by their gcd′s, i.e., we consider the basis $f_1 = \begin{bmatrix} \frac{\alpha}{\gcd(\alpha,\gamma)} \\ \frac{\gamma}{\gcd(\alpha,\gamma)} \end{bmatrix}$, $f_2 = \begin{bmatrix} \frac{-1+\alpha}{\gcd(1-\alpha,\gamma)} \\ \frac{\gamma}{\gcd(1-\alpha,\gamma)} \end{bmatrix}$. Finally, we take $P = \begin{bmatrix} \frac{\alpha}{\gcd(\alpha,\gamma)} & \frac{-1+\alpha}{\gcd(1-\alpha,\gamma)} \\ \frac{\gamma}{\gcd(\alpha,\gamma)} & \frac{\gamma}{\gcd(1-\alpha,\gamma)} \end{bmatrix}$ and so

$$\det(P) = \frac{\alpha\gamma + (1-\alpha)\gamma}{\gcd(\alpha,\gamma)\gcd(1-\alpha,\gamma)} = \frac{\gamma}{\gamma} = 1,$$

since $\gcd(\alpha, 1-\alpha) = 1$ implies

$$\gcd(\alpha,\gamma)\gcd(1-\alpha,\gamma) = \gcd(\alpha(1-\alpha),\gamma) = \gcd(\beta\gamma,\gamma) = \gamma.$$

For the matrix V which, for given u, v, satisfies $\det(V) = u$ and $\mathrm{Tr}(EV) = v$, in the $\{f_1, f_2\}$ basis, we can take $V' = \begin{bmatrix} v & -1 \\ u & 0 \end{bmatrix}$, that is, $w(f_1) = vf_1 - f_2$, $w(f_2) = uf_1$, if we denote by w the endomorphism associated to V'.

Finally, we have to come back to the standard basis for w and so

$$V = PV'P^{-1} = PV'adj(P)$$

as $\det(P) = 1$. To simplify the notations let $s = \gcd(\alpha, \gamma)$ and $q = \gcd(1-\alpha, \gamma)$. Note that s and q are chosen (to give $\det(P) = 1$) such that $sq = \gamma$. Hence

$$V = \begin{bmatrix} \alpha v - \dfrac{(1-\alpha)su}{q} + \dfrac{\alpha q}{s} & \beta v - \dfrac{(1-\alpha)^2 u}{q^2} - \dfrac{\alpha^2}{s^2} \\ \gamma v + s^2 u + q^2 & (1-\alpha)v + \dfrac{(1-\alpha)su}{q} - \dfrac{\alpha q}{s} \end{bmatrix}.$$

Case 2. Assume $\gamma = 0$ and $\beta \neq 0$. Then $\alpha \in \{0,1\}$ and it suffices to deal with $E = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}$, obtaining the other subcase by conjugation with $E_{12} + E_{21}$. Now we use the other possible choice for a basis (i.e., $\begin{bmatrix} \beta \\ 1-\alpha \end{bmatrix}$ and $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$) and finally choose $f_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $f_2 = \begin{bmatrix} -\beta \\ 1 \end{bmatrix}$, so $P = \begin{bmatrix} 1 & -\beta \\ 0 & 1 \end{bmatrix}$ and

$$V = PV'P^{-1} = \begin{bmatrix} v+\beta & \beta v + \beta^2 + u \\ -1 & -\beta \end{bmatrix}.$$

.

Case 3. Assume $\beta = \gamma = 0$. Up to conjugation we have $E = E_{11}$. We can take

$$V = V' = \begin{bmatrix} v & -1 \\ u & 0 \end{bmatrix},$$

as $\det(V) = u$ and $\mathrm{Tr}(E_{11}V) = v$. □

What follows is a generalization of the previous lemma. Recall that a ring is called *Bézout* if the sum of two principal ideals is also a principal ideal, meaning Bézout's identity holds for every pair of elements. A ring is termed (right) *Hermite* (in the sense of Kaplansky) if, for any two elements a and b of the ring, there exists an element d and an invertible 2×2 matrix M over the ring such that $\begin{bmatrix} a \\ b \end{bmatrix} M = \begin{bmatrix} d \\ 0 \end{bmatrix}$.

Recall that every elementary divisor domain (EDD) is a Hermite domain, and every Hermite domain is a Bézout domain.

We begin with a simple result.

Lemma 2.12. *Let* $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *be a* 2×2 *matrix over a Bézout domain* R. *Then* $\gcd(B) = 1$ *iff there exists a matrix* M *such that* $\mathrm{Tr}(MB) = 1$.

Proof. $\gcd(B) = 1$ iff there are elements $a, b, c, d \in R$ such that

$$a\alpha + b\gamma + c\beta + d\delta = 1.$$

But this is precisely $(\mathrm{Tr}(BM) =)$ $\mathrm{Tr}(MB) = 1$ with $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. $\square$

Next, recall from [3] the following

Definition (**1.10** in [3]) We say that R is a $WJ_{2,1}$ *ring* if for each unimodular row $(\alpha, \beta, \gamma, \delta) \in R^4$ and every $(u, v) \in R^2$, there exists $(x, y, z, w) \in R^4$ such that $\alpha x + \beta z + \gamma y + \delta w = v$ and $xw - yz = u$. Equivalently, for every $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $\gcd(B) = 1$, there exists $V = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ such that $\mathrm{Tr}(BV) = v$, $\det(V) = u$.

The definition of the class of rings $J_{2,1}$ (defined as $\mathrm{WJ}_{2,1}$ and Hermite), first appeared in [8], in a different form (we don't recall here). Also in [3], it is proved that these two definitions are equivalent (see Proposition 7.1). We also mention that in [8], it is proved that *every elementary divisor ring is* $J_{2,1}$ (see Proposition 4.8).

Even more general we have the following

Definition. R is a $WWJ_{2,1}$ *ring* if for each unimodular row $(\alpha, \beta, \gamma, \delta) \in R^4$ with $\alpha\delta = \beta\gamma$ and every $(u, v) \in R^2$, there exists $(x, y, z, w) \in R^4$ such that $\alpha x + \beta z + \gamma y + \delta w = v$ and $xw - yz = u$. Equivalently, for every $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $\gcd(B) = 1$ and $\det(B) = 0$, there exists $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $\mathrm{Tr}(BV) = v$, $\det(V) = u$.

Therefore we have the following sequence of implications

$$EDR \Rightarrow J_{2,1} \Rightarrow WJ_{2,1} \Rightarrow WWJ_{2,1}.$$

Finally, in [3], it is proved that a $J_{2,1}$ domain is EDD (i.e., elementary divisor domain). Thus, for (commutative) domains, $J_{2,1}$ is equivalent to EDD.

By the above we have the following generalization of Lemma 2.11.

Proposition 2.13. *Let R be an EDD and let B be a 2×2 matrix over R, with* $\det(B) = 0$, $\gcd(B) = 1$ *and $u, v \in R$. There exists a 2×2 matrix V such that* $\det(V) = u$ *and* $\mathrm{Tr}(BV) = v$.

Over Bézout domains, the next result is a necessary condition for ssr1 of 2×2 matrices.

Theorem 2.14. *Let R be a Bézout domain and let $A \in \mathbb{M}_2(R)$. Then $ssr(A) = 1$ only if* $\det(A) = 0$ *or* $\det(A) \in U(R)$.

Proof. Suppose $ssr(A) = 1$, i.e., for every X there is a Y such that $A^2 + Y(XA - I_2)$ is invertible. Choosing $X = r \cdot adj(A)$ for any $r \in R$, we get $Y(XA - I_2) = Y(r\det(A) - 1)I_2$ and so there is Y such that

$$\det[A^2 + Y(r\det(A) - 1)I_2] = \det[A^2 + (r\det(A) - 1)Y] \in U(R).$$

By computation, for $A = [a_{ij}]$ we have

$$A^2 + (r\det(A) - 1)Y = \left[\begin{array}{cc} a_{11}^2 + a_{12}a_{21} + (r\det(A) - 1)x & a_{12}\mathrm{Tr}(A) + (r\det(A) - 1)y \\ a_{21}\mathrm{Tr}(A) + (r\det(A) - 1)z & a_{12}^2 a_{21} + a_{22}^2 + (r\det(A) - 1)t \end{array}\right],$$

which gives $\det^2(A) + (r\det(A) - 1)\gamma + (r\det(A) - 1)^2\det(Y)$ where

$$\gamma = (a_{12}a_{21} + a_{22}^2)x - \mathrm{Tr}(A)(a_{21}y + a_{12}z) + (a_{11}^2 + a_{12}a_{21})t.$$

Denoting $\alpha = \det(A)$ and $\beta = \gamma + (r\det(A) - 1)\det(Y)$, the condition becomes

$$\alpha^2 + (r\alpha - 1)\beta \in U(R).$$

For every r, these quadratic Diophantine equations should be solvable. In particular, the equations $\alpha^2 + (r\alpha - 1)\beta = 1$ should be solvable for every r.

For $r = 1$ we get $(\alpha - 1)(1 - \beta) = 0$ whence (over any domain), $\alpha = 1$ or $\beta = 1$.

For $r = 0$ we get $\alpha^2 = 1 - \beta$ so that if $\beta = 1$, $\alpha = 0$ follows.

Therefore $\det(A) = 0$ or $\det(A) \in U(R)$. □

Consequently, using Theorem 2.10 and Proposition 2.13, we have the final structure result for zero determinant matrices which have the ssr1 property.

Theorem 2.15. *Let A be any nonzero 2×2 matrix over an EDD, $\det(A) = 0$, $\mathrm{Tr}(A) \neq 0$ and for $\delta = \gcd(A)$, write $A = \delta B$. Then $ssr(A) = 1$ iff $\mathrm{Tr}(B) = 1$ (equivalently, if B is idempotent) or, all prime divisors of $\mathrm{Tr}(B)$ divide δ.*

3. Applications

3.1. Ring multiples of idempotent matrices.

Lemma 2.11 has an important consequence.

Theorem 3.1. *Over any Bézout domain R, let $r \in R$ and $E \in \mathbb{M}_2(R)$ be a nontrivial idempotent. Then $ssr(rE) = 1$.*

Proof. Let $E = \begin{bmatrix} \alpha & \beta \\ \gamma & 1-\alpha \end{bmatrix}$ be a nontrivial idempotent, i.e., $\alpha(1-\alpha) = \beta\gamma$ and let $r \in R$. As $\det(E) = 0$ we also have $\det(rE) = 0$ and we use Corollary 2.4 for $A = rE$. We have

$$\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(A)\mathrm{Tr}(Aadj(Y)) =$$
$$\det(Y)(1 - r\mathrm{Tr}(XE)) - r^2\mathrm{Tr}(Eadj(Y)) = 1,$$

which is solvable for $\det(Y)$ and $\mathrm{Tr}(Eadj(Y))$ by the Bézout hypothesis, since $\gcd(1 - r\mathrm{Tr}(XE), r^2) = 1$.

If $u(1 - r\mathrm{Tr}(XE)) - vr^2 = 1$, it only remains to choose a square unitizer Y such that $\det(Y) = u$ and $\mathrm{Tr}(Eadj(Y)) = v$.

We can simplify this as follows. It is well-known that $\det(adj(Y) = \det(Y)^{n-1}$ and so for $n = 2$, $\det(adj(Y)) = \det(Y)$. Hence we can search for a matrix V such that $\det(V) = u$ and $\mathrm{Tr}(EV) = v$. Having found $V = adj(Y)$, we come back to Y using $adj(adj(Y)) = Y$, i.e., $Y = adj(V)$.

Such a matrix V exists by the Lemma 2.11 and we are done. □

Note that *ring multiples of invertible matrices* may not have the ssr1 property. An example is $2I_2$ over $\mathbb{Z}$ (see Proposition 3.14, Section 3).

3.2. **Some square unitizers found using Lemma 2.11.**

Examples. 1) Let $A = 2\begin{bmatrix} 3 & 1 \\ -6 & -2 \end{bmatrix}$ be the double of a nontrivial integral idempotent matrix. With the previous notations, we have $r = 2$ and $\alpha = 3$, $\beta = 1$, $\gamma = -6$ and we choose $s = \gcd(\alpha, \gamma) = 3$, $q = \gcd(1-\alpha, \gamma) = -2$ (to have $sq = \gamma$). For $X = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ (i.e., $a = b = 2$, $c = 3$, $d = 4$), we get $V = \begin{bmatrix} -17 & -6 \\ 31 & 11 \end{bmatrix}$ and $Y = \begin{bmatrix} 11 & 6 \\ -31 & -17 \end{bmatrix}$. Indeed, $A^2 + Y(XA - I_2) = \begin{bmatrix} -311 & -106 \\ 889 & 303 \end{bmatrix}$ has determinant 1.

2) Let $A = 2\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. With the previous notations, we have $r = 2$ and $\alpha = \beta = 1$, $\gamma = 1 - \alpha = 0$. Hence, we are in Case 2 of the Lemma 2.11.

For $X = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}$ (i.e., $a = b = 2$, $c = 3$, $d = 4$), we get $V = \begin{bmatrix} 3 & 2 \\ -1 & -1 \end{bmatrix}$. Indeed, for $Y = adj(V)$,

$$A^2 + Y(XA - I_2) = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} -11 & -10 \\ 21 & 19 \end{bmatrix}$$

has determinant 1.

As we can see from the above examples, the square unitizers provided by Theorem 3.1 may not be simple. Of course, in general the square unitizers (if any) are not unique. Next, we **list** some multiples of idempotent matrices together with some simple square unitizers.

(i) The matrices $A_{r,s} := \begin{bmatrix} r & rs \\ 0 & 0 \end{bmatrix}$ over any ring R with $r, s \in R$. Replacing in Corollary 2.4 gives $\det(Y)[1 - r(a + cs)] - r^2(t - sz) \in U(R)$ (for $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$) . Choosing $\det(Y) = 1 + r(a + cs)$ and $t - sz = -(a + cs)^2$ yields

$$Y = \begin{bmatrix} 0 & -1 \\ 1 + r(a + cs) & s[1 + r(a + cs)] - (a + cs)^2 \end{bmatrix},$$

a suitable square unitizer. For $s = 0$ we obtain a square unitizer for rE_{11} (for rE_{22} this follows by conjugation with the involution $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$) and for $s = 1$, a square unitizer for $\begin{bmatrix} r & r \\ 0 & 0 \end{bmatrix}$.

(ii) The integral matrices $A_{2,n} = \begin{bmatrix} 2 & 2n \\ 0 & 0 \end{bmatrix}$ for any integer n. As before, denote by $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ an arbitrary matrix of $\mathbb{M}_2(\mathbb{Z})$.

If $a+cn$ is even, we can choose the square unitizer $Y = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2}(a + cn) \end{bmatrix}$ and if $a+cn$ is odd, we can choose the square unitizer $Y = \begin{bmatrix} 0 & 1 \\ 3 & \frac{3}{2}(a + cn - 1) + 1 \end{bmatrix}$.

Remarks. If r divides s (as in (i) or (ii) above) then A is a multiple of an idempotent matrix and so has ssr1 by Theorem 3.1. At this point, one could state two possible converses.

(a) If a multiple $B = rA$ of a matrix has ssr1, then A has ssr1.

(b) If a multiple $B = rA$ of a matrix has ssr1, then A is idempotent.

None of these holds, as witnessed by $B = \begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_2(\mathbb{Z})$. That $ssr(B) = 1$, follows from Theorem 3.4 and the next section.

There is a class of rings over which the proof of Theorem 3.1 becomes straightforward.

Following Steger [9], we define a ring R as an *ID* ring if every idempotent matrix over R is similar to a diagonal matrix. Examples of ID rings include: division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings and serial rings.

Over an ID ring, any nontrivial idempotent 2×2 matrix is similar to E_{11} and thus the proof simplifies to showing that ring multiples of E_{11} have the ssr1 property. This has already been addressed in the previous discussion (see the list, (i)), for any (commutative) ring.

3.3. **Matrices with a zero row (or zero column).**

As an application of the previous section, we examine the 2×2 matrices over Bézout domains that have at least one zero row or one zero column. By applying conjugation and/or transposition, it suffices to focus on 2×2 matrices with a zero second row, specifically $M_{r,s} := \begin{bmatrix} r & s \\ 0 & 0 \end{bmatrix}$ where $r, s \in R$.

Over Bézout domains, we provide a characterization of the $M_{r,s}$ matrices for which the XY-equation is solvable.

First, we discard some special cases.

The matrices $M_{0,s}$ with $s \notin J(R)$ do not have the ssr1 property over reduced rings (see Example 2.7). The matrices $M_{r,0}$ with $r \neq 0$ are multiples of idempotents and thus possess the ssr1 property (by Theorem 3.1). Naturally, $M_{0,0} = 0_2$ has the ssr1 property, as it is idempotent.

Lemma 3.2. *Over any ring R, if $r \in U(R)$ then $ssr(M_{r,s}) = 1$ for every s.*

Proof. If r is a unit then a square unitizer is $Y = \begin{bmatrix} 0 & 0 \\ 0 & r^{-1} \end{bmatrix}$ (independent of X). Indeed, in this case $\det(Y) = 0$ and Corollary 2.4 reduces to $r \cdot 1 = r \in U(R)$. □

Proposition 3.3. *Let R be a Bézout ring. If* $\gcd(r, s) = 1$ *and $r \notin U(R)$ then $ssr(M_{r,s}) \neq 1$.*

Proof. If $ar + bs = 1$, we can take $X = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$. For this choice, it follows that $\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(A)\mathrm{Tr}(Aadj(Y))$ is a multiple of r. □

Theorem 3.4. *Let R be a Bézout domain, $r, s \in R$ and $r, s \neq 0$. The following conditions are equivalent.*

(i) The XY-equation of $M_{r,s}$ is solvable,

(ii) $\gcd(sc-1,r)=1$ *for every* $c\in R$.
Let $r=\delta r_1$, $s=\delta s_1$ *with* $\delta=\gcd(r,s)$. *The conditions are equivalent to*
(iii) Every prime divisor of r_1 *divides* δ.

Proof. We can discard the case $r=s$. As a multiple of an idempotent, it has ssr1. On the other side, $\gcd(rc-1,r)=1$ for every $c\in R$ (see also a square unitizer, in the list given in the previous subsection).

So is the case $r\in U(R)$, when $M_{r,s}=rM_{1,r^{-1}s}$ (see Lemma 3.2, on LHS, and $\gcd(sc-1,r)=1$ for every $c\in R$, on RHS). Therefore we assume $r\notin U(R)$. By Proposition 3.3 above, if $\gcd(r,s)=1$ then $ssr(M_{r,s})\neq 1$.

Hence we further assume $\gcd(r,s)=\delta\notin U(R)$ (so $\delta\neq 1$, modulo association). If $r=\delta r_1$, $s=\delta s_1$, like before, we can discard the case $r_1\in U(R)$, when again $ssr(M_{r,s})=1$.

So we are left with the case when both δ and r_1 are not units and $\gcd(r_1,s_1)=1$.

(i) $\Leftrightarrow$ (ii) The XY-equation is now

$$\det(Y)(\delta r_1a+\delta s_1c-1)+\delta^2r_1(r_1t-s_1z)=1.$$

It is solvable for $\det(Y)$ and $\mathrm{Tr}(M_{r_1,s_1}adj(Y))$ iff $\gcd(\delta r_1a+\delta s_1c-1,\delta^2r_1)=\gcd(\delta s_1c-1,\delta^2r_1)=1$ for every $c\in R$.

As $\gcd(\delta s_1c-1,\delta^2)=\gcd\delta s_1c-1,\delta)=1$, the XY-equation is solvable iff $\gcd(\delta s_1c-1,r_1)=1$ iff $\gcd(sc-1,r)=1$.

(ii) $\Leftrightarrow$ (iii) By Lemma 2.9, the condition is equivalent to the prime divisors of r_1 dividing δs_1. Since $\gcd(r_1,s_1)=1$ this is equivalent to (iii). $\square$

For the following result the Bézout domain hypothesis is not necessary.

Proposition 3.5. *Let* R *be a ring,* $r\notin U(R)$ *and* $A=M_{r,rk+1}$ *for some* $k\in R$. *Then* $ssr(A)\neq 1$ *for any* $k\in R$.

Proof. Since $\det(A)=0$, taking $X=E_{21}$ and replacing in Corollary 2.4, in order to have $ssr(A)=1$, $\det(Y)(-rk)-r[rt-(rk+1)z]=r\beta$ should be unit. But it is not, as $r\notin U(R)$ (and R is commutative). $\square$

Remark. Special cases include $k=1$, i.e., $ssr\begin{bmatrix} r & r+1\\ 0 & 0\end{bmatrix}\neq 1$, and $k=0$, i.e., $ssr\begin{bmatrix} r & 1\\ 0 & 0\end{bmatrix}\neq 1$, whenever $r\notin U(R)$.

Since $\begin{bmatrix} 1 & r\\ 0 & 0\end{bmatrix}$ is idempotent, it has ssr1. Hence $ssr(M_{r,s})=1$ does not imply (in general) $ssr(M_{s,r})=1$.

If r, s are coprime then $M_{r,s}$ is unit regular (see [5]) and so, has sr1. However, this condition is not sufficient for ssr1: the nonexamples above are unit-regular but have not ssr1. Over any (commutative) ring we also have the following nonexample.

Proposition 3.6. *Let R be a ring. Then ssr* $\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} \neq 1$ *whenever* $3 \notin U(R)$.

Proof. Take $X = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$. Since $\det(A) = 0$, it follows that
$\det(Y)(1 - \mathrm{Tr}(XA)) - \mathrm{Tr}(A)\mathrm{Tr}(Aadj(Y)) = 3(2\det(Y) - 3t + 5z) \notin U(R)$. □

We close this section with some more examples.

Examples. 1) *The positive multiples of* $M_{2,3}$*, as integral matrices.* We have $ssr(2nM_{2,3}) = 1$ and $ssr((2n+1)M_{2,3}) \neq 1$.

Using Theorem 2.10, we have $\mathrm{Tr}(B) = 2 \mid 2n$ but $\mathrm{Tr}(B) = 2 \nmid 2n+1$. In the positive case, for $n = 1$, an explicit square unitizer is given in the next section. For the general n case, providing an explicit square unitizer is just sketched, also in the next section.

2) *Ssr1 has not "the complementary" property*, that is, $ssr(A) = 1$ does not imply $ssr(I_2 - A) = 1$.

Indeed, take the unit $U = \begin{bmatrix} -1 & -3 \\ 0 & 1 \end{bmatrix}$ which has ssr1. Then $I_2 - U = M_{2,3}$ has not ssr1.

While we already mentioned that $ssr(M_{r,s}) = 1$ does (in general) not imply $ssr(M_{s,r}) = 1$, there are examples when both have ssr1.

Proposition 3.7. *Let p be a prime number and k, l some positive integers. Then* $ssr(M_{p^k,p^l}) = 1$.

Proof. Indeed, if $k \leq l$ then M_{p^k,p^k} is a multiple of an idempotent and so (by Theorem 3.1) has ssr1. For $k \leq l$, as $p^k \mid p^l$, a square unitizer is given as in (i), the previous list.

If $k > l$, the XY-equation is

$$(p^k a + p^l c - 1)\det(Y) + p^{k+l}(p^{k-l}t - z) = 1.$$

Denoting $m = p^k a + p^l c - 1$, we successively get $m(xt - yz) + p^{k+l}(p^{k-l}t - z) = 1$ and $(mx + p^{2k})t - (my + p^{k+l})z = 1$. Now choose $x = (p^{k-l} - 1)y$. Then

$$\gcd((p^{k-l} - 1)my + p^{2k}, my + p^{k+l}) = \gcd(p^{k+l}, my + p^{k+l}) = \gcd(p^{k+l}, my) = 1,$$

if we choose any y coprime to p (as m is also coprime to p).

For $y = 1$ and consequently $x = p^{k-l} - 1$, we have a square unitizer of form $Y = \begin{bmatrix} p^{k-l} - 1 & 1 \\ z & t \end{bmatrix}$ with z, t given by the solvable Diophantine equation

$$[(p^{k-l} - 1)m + p^{2k}]t - (m + p^{k+l})z = 1.$$

Now denote $n = m + p^{k+l}$. Then

$$n[(p^{k-l} - 1)t - z] + p^{k+l}t = 1.$$

Since $\gcd(n, p) = 1$, this equation is solvable and if $n\alpha + p^{k+l}\beta = 1$ (i.e., (α, β) is any particular solution), it follows that $t = \beta$, $z = (p^{k-l} - 1)\beta - \alpha$. Finally $Y = \begin{bmatrix} p^{k-l} - 1 & 1 \\ (p^{k-l} - 1)\beta - \alpha & \beta \end{bmatrix}$ is a square unitizer for M_{p^k,p^l}. □

3.4. **Integral matrices.**

As previously mentioned, the ssr1 property is not invariant under equivalences. Therefore, when characterizing ssr1 integral matrices, we cannot rely on diagonal reduction (or the Smith canonical form), which applies to elementary divisor rings.

However, as stated in Lemma 2.1 (ii), ssr1 is invariant under similarities. This allows us to use diagonalization, based on eigenvalues and eigenspaces, although its applicability is limited.

The limitation arises because many 2×2 integral matrices possess the ssr1 property but are not diagonalizable (e.g., $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ which has no real eigenvalues but satisfies ssr1, as it is invertible).

We also note that for every matrix A, both A and $-A$ either have or lack the ssr1 property. Hence, for zero determinant matrices, we can assume without loss of generality that the trace is positive (by Corollary 2.6, a zero trace corresponds only to the zero matrix).

Recall from [2] the following characterization.

Theorem 3.8. *For any matrix $A \in \mathbb{M}_2(\mathbb{Z})$, the following conditions are equivalent.*

(i) A is left strongly regular, i.e., there exists $B \in \mathbb{M}_2(\mathbb{Z})$ such that $A^2B = A$;

(ii) A is strongly regular, i.e., there exists $B \in \mathbb{M}_2(\mathbb{Z})$ such that $A^2B = A = BA^2$;

(iii) A is a unit or an idempotent or a minus idempotent.

That these matrices have also ssr1, was already clear in the Introduction (and Lemma 2.1, (iii)). By Theorem 3.1, we already know that the set of 2×2 integral ssr1 matrices is considerably larger as it contains (at least) all integer multiples of idempotent matrices.

The examples discussed in the previous section show that the units and the integer multiples of idempotents *do not exhaust* the set of all the integral 2×2 matrices which have ssr1.

Theorem 2.14 has the following consequence.

Corollary 3.9. *Let $A \in \mathbb{M}_2(\mathbb{Z})$. Then $ssr(A) = 1$ only if* $\det(A) \in \{\pm 1, 0\}$.

Next, according to Theorem 2.15, we have the characterization of integral 2×2 matrices which have ssr1.

Theorem 3.10. *Let A be any nonzero 2×2 integral matrix,* $\det(A) = 0$, $\mathrm{Tr}(A) > 0$ *and for $\delta = \gcd(A)$, write $A = \delta B$. Then $ssr(A) = 1$ iff $\mathrm{Tr}(B) = 1$ (equivalently, if B is idempotent) or, all prime divisors of $\mathrm{Tr}(B)$ divide δ.*

Some special cases include

Corollary 3.11. *If $\mathrm{Tr}(B)$ is even and δ is odd then $ssr(A) \neq 1$.*

Proof. Follows from the characterization as 2 does not divide δ. $\square$

Corollary 3.12. *If $\delta = 1$ and $\mathrm{Tr}(B) \neq 1$ then $ssr(A) \neq 1$.*

and

Proposition 3.13. *The matrices $M_{uv} = \begin{bmatrix} 1 & u \\ v & uv \end{bmatrix}$ have sr1 over any ring (see* [1]*) but not ssr1 unless $1 + uv$ is a unit.*

Proof. Since $\det(A) = 0$, we have $A^2 = \mathrm{Tr}(A)A = (1 + uv)A$. Moreover, for $X = E_{11}$ we get $XA = \begin{bmatrix} 1 & u \\ 0 & 0 \end{bmatrix}$ so $1 - \mathrm{Tr}(XA) = 0$. Replacing in Corollary 2.4, we obtain that the XY-equation of M_{uv} is solvable only if

$$\mathrm{Tr}(A^2 adj(Y)) = (1 + uv)Tr(Aadj(Y)) \in U(R).$$

Over any (commutative) ring, a necessary condition is $1 + uv \in U(R)$. $\square$

Over the integers, this occurs iff $uv = 0$ or else $uv = -2$. The first case requires $u = 0$ or $v = 0$, and the second $u, v \in \{\pm 1, \pm 2\}$. In the first case we have idempotent matrices, which are known to have ssr1, while in the second, we just have four matrices: $\begin{bmatrix} 1 & \pm 1 \\ \mp 2 & -2 \end{bmatrix}$ and their transposes. For these four it suffices to discuss $\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$. Using Lemma 2.1, (iii) and (ii), this matrix has ssr1 iff $-\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$ has ssr1. But this holds as the latter is an idempotent matrix.

Some other examples include

Proposition 3.14. *An integral scalar matrix $A = nI_2$ has ssr1 iff $n \in \{-1, 0, 1\}$.*

Proposition 3.15. *An integral diagonal matrix $A = \begin{bmatrix} n & 0 \\ 0 & m \end{bmatrix}$ has ssr1 iff $n = 0$ and m is arbitrary, or $m = 0$ and n is arbitrary, or $n, m \in \{-1, 0, 1\}$.*

Equivalently, the only 2×2 ssr1 diagonal matrices are: 0_2, $\pm I_2$, $\pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, nE_{11}, mE_{22} for any integers n, m. That is, the units and the multiples of nontrivial idempotents.

3.5. **Jacobson Lemma for ssr1.**

Jacobson's Lemma holds for sr1 elements (see Theorem 6.10, [6]), meaning that for any elements a, b in a ring R, we have $sr(1 - ab) = 1$ iff $sr(1 - ba) = 1$.

In this final subsection we address the following natural question: Does Jacobson's Lemma hold for ssr1 elements ?

The proof for sr1 elements relies on elementary row and column operations, as well as row switches, since these operations preserve sr1 (being equivalences). However, such an approach is not feasible for ssr1, as ssr1 is neither invariant under equivalences nor under elementary operations (see also the beginning of the next section).

By *Jacobson's Lemma for ssr1* in a ring R we mean:

$$ssr(1 - ab) = 1 \Longrightarrow ssr(1 - ba) = 1$$

for all elements a, b of the ring R.

We begin by mentioning a statement that is strongly supported by computational evidence.

Conjecture 3.16. *For 2×2 integral matrices A, B, assume $\det(I_2 - AB) = 0$. Then $\gcd(I_2 - AB) = \gcd(I_2 - BA)$.*

Note that $\mathrm{Tr}(I_2 - AB) = \mathrm{Tr}(I_2 - BA)$, for any matrices A, B over any ring. Furthermore, if $\det(I_2 - AB) = 0$ then $\det(I_2 - BA) = 0$, as well.

As an application of our results on ssr1 for integral 2×2 matrices, we can demonstrate the following result.

Proposition 3.17. *If the conjecture above holds then the Jacobson Lemma for ssr1 holds for integral* 2×2 *matrices.*

PROOF. First recall (see Theorem 3.9) that only matrices A with $\det(A) \in \{\pm 1, 0\}$ have ssr1 in $\mathbb{M}_2(\mathbb{Z})$.

Next, suppose $ssr(I_2 - AB) = 1$ with $AB \neq 0_2, I_2$.

According to this theorem, we have two possible cases:

Case 1. $C := I_2 - AB$ is a unit.

Then using the (classical) Jacobson's Lemma, $D := I_2 - BA$ is also a unit, so has ssr1.

Case 2. $\det(I_2 - AB) = 0$.

Note that

$$\det(I_2 - AB) = \det(AB - I_2) = \det(A)\det(B) - \mathrm{Tr}(AB) + 1 = \det(I_2 - BA),$$

as $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$. Hence also $\det(I_2 - BA) = 0$.

Further, clearly, $\mathrm{Tr}(C) = \mathrm{Tr}(D)$ so actually C and D both have zero determinant and the same trace. Therefore, if the conjecture above holds, the claim follows from Theorem 2.10. $\square$

The question of whether the Jacobson Lemma for ssr1 holds in more general rings remains unsolved. Even more ambitiously, whether the Super Jacobson Lemma for ssr1 applies to a broader class of rings also remains an open question.

4. Finding square unitizers for $\begin{bmatrix} 4 & 6 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

As mentioned in the previous section, in the sequence, $M_{2,3}$, $2M_{2,3}$, $3M_{2,3}$, $4M_{2,3}$,... the odd multiples do not possess ssr1. However, for the even multiples, the XY-equation is solvable. In this section, we provide explicit square unitizers for $2M_{2,3}$, this way showing that $ssr(2M_{2,3}) = 1$.

Note that $ssrM_{6,4} \neq 1$ follows from Theorem 3.4, as $\gcd(6, 1-4) = 3 \neq 1$. Hence, as previously mentioned, $ssrM_{r,s} = 1$ does not generally imply that $ssr(M_{s,r}) = 1$.

In other words, ssr1 for 2×2 matrices is not invariant under *column switches*. Moreover, it is also not invariant to *elementary row (or column*, by transposition) *operations*. For example, $M_{1,1}$ being idempotent, has ssr1. However, subtracting the first row from the second results in a nonzero nilpotent matrix, which does not have ssr1.

4.1. Explicit square unitizers of $M_{4,6}$.

As $\det(M_{4,6}) = 0$ and $\mathrm{Tr}(M_{4,6}) = 4$, we start by replacing $M_{4,6}$ in Corollary 2.4. The XY-equation is now

$$\det(Y)(4a + 6c - 1) + 8(2t - 3z) = 1 \qquad (*).$$

As for every $a, c \in \mathbb{Z}$, $4a + 6c - 1$ is odd, so coprime with 8, (*) is a linear Diophantine equation in 2 variables, namely $\det(Y)$ and $2t - 3z$. Denote $k = 2a + 3c$, $s = \det(Y)$, $w = 2t - 3z$. Then (*) becomes

$$(2k - 1)s + 8w = 1$$

and browsing the (minimal) Bézout coefficients, as for s, these appear repeatedly modulo 4. More precisely,

if $k = 4l$ we can choose $s = -1$, $w = l$
if $k = 4l + 1$ we can choose $s = 1$, $w = -l$
if $k = 4l + 2$ we can choose $s = 3$, $w = -3l - 1$
if $k = 4l + 3$ we can choose $s = -3$, $w = 3l + 2$.

In each of the 4 cases above, we choose a square unitizer $Y = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$ with $\det(Y) = s$, $2t - 3z = w$.

Case 1. We are searching for integers x, y, z, t such that $\det(Y) = xt - yz = -1$ and $2t - 3z = l$ for any given integer l.

As $\gcd(2, 3) = 1$ the linear Diophantine equation $2t - 3z = l$ has the general solution $t = 2l - 3k$, $z = l - 2k$ for some integer k.

Replacing, we can write

$$x(2l - 3k) - y(l - 2k) = -1.$$

If we choose k coprime with l, it is readily seen that $2l - 3k$, $l - 2k$ are also coprime. For any particular solution (x_0, y_0), the general solution of this linear Diophantine equation is $x = x_0 + (l - 2k)m$, $y = y_0 + (2l - 3k)m$, for some m. For $k = 1$ we choose $(x_0, y_0) = (-1, -2)$ and so for $m = 0$ we finally obtain a square unitizer $Y = \begin{bmatrix} -1 & -2 \\ l - 2 & 2l - 3 \end{bmatrix}$ with $\det(Y) = -1$ and $2t - 3z = l$. As for a and c, here $2a + 3c \equiv 0 \pmod 4$ and $l = \dfrac{2a + 3c}{4}$.

Case 2. We are searching for integers x, y, z, t such that $\det(Y) = xt - yz = 1$ and $2t - 3z = -l$, for any given integer l. Similar to Case 1 (by changing the signs of the entries in the second row) we obtain $Y = \begin{bmatrix} -1 & -2 \\ -l+2 & -2l+3 \end{bmatrix}$ with $\det(Y) = 1$ and $2t - 3z = -l$. Here $2a + 3c \equiv 1 \pmod 4$ and $l = \dfrac{2a+3c-1}{4}$.

Case 3. We are searching for integers x, y, z, t such that $\det(Y) = xt - yz = 3$ and $2t - 3z = -3l - 1$ for any given integer l. The general solution for $2t - 3z = -3l - 1$ is now $t = -2(3l+1) - 3k$, $z = -3l - 1 - 2k$, for some integer k. As in Case 1 we can choose $k = 1$ (coprime with $3l + 1$) and we are looking for a particular solution of

$$-x(6l+5) + 3y(l+1) = 3.$$

Note that $\gcd(6l+5, 3) = 1 = \gcd(6l+5, l+1)$ and so $\gcd(6l+5, 3(l+1) = 1$. A particular solution is $x = 3$, $y = 6$ and so $Y = \begin{bmatrix} -3 & -6 \\ 3l & 6l+1 \end{bmatrix}$ is a square unitizer. Here $2a + 3c \equiv 2 \pmod 4$ and $l = \dfrac{2a+3c-2}{4}$.

Case 4. Finally, we are searching for integers x, y, z, t such that $\det(Y) = xt - yz = -3$ and $2t - 3z = 3l + 2$ for any given integer l. Similar to Case 3 we obtain $Y = \begin{bmatrix} 3 & 6 \\ -3l-3 & -6l-5 \end{bmatrix}$, a square unitizer. Here $2a + 3c \equiv 3 \pmod 4$ and $l = \dfrac{2a+3c-3}{4}$.

Remark. For $M_{4n,6n}$ for some (positive) integer n, the XY-equation is

$$\det(Y)[2n(2a+3b) - 1] + 8n^2(2t - 3z) = 1$$

with coprime $2n(2a+3b) - 1$ and $8n^2$. Hence it is solvable for $\det(Y)$ and $2t - 3z$. We continue as above, not modulo 4 but modulo $2n$.

4.2. **Explicit square unitizers for even multiples of $S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.**

Proposition 4.1. *For the integer multiples of S, we have*
$ssr(kS) = \begin{cases} = 1 & if \quad k \text{ is even} \\ \neq 1 & if \quad k \text{ is odd} \end{cases}$.

Proof. According to Theorem 2.10, the negative (odd k) statement follows as $\mathrm{Tr}(S) = 2$ and $\delta = k$. For the positive one (even k), in order not to lengthen this exposition, some square unitizers are just sketched below.

For $ssr(2S) = 1$.

As $\det(S) = 0$ and $S^2 = 8S$, Corollary 2.4 requires

$$\det(Y)(1 - 2(a+b+c+d)) - 8(t - z - y + x) = -1,$$

equation which we write $(2k-1)s+8w=1$. This equation already appeared above, splitted into four cases corresponding to the reminder of dividing $k=a+b+c+d$ by 4.

Case 1, that is $k=4l$. We are searching for $\det(Y)=-1$, $x-y-z+t=l$. The solution is simpler because we can choose y,z,t as parameters and so $x=l+y+z-t$. Replacing into $xt-yz=-1$ and choosing $t=1$, we find the square unitizer $Y=\begin{bmatrix} -1 & 0 \\ -l & 1 \end{bmatrix}$.

The other three cases are analogous.

For $ssr(4S)=1$.

Now $S^2=32S$ and denoting $k=a+b+c+d$, for Corollary 2.4 we require $(4k-1)\det(Y)+32(t-z-y+x)=1$.

Case 1, that is $k=4l$. We have to further divide into 2 cases.

If l is even, we are searching for $\det(Y)=-1$ and $t-z-y+x=\frac{l}{2}$.

If l is odd, we are searching for $\det(Y)=\frac{l-1}{2}$ and $t-z-y+x=\frac{15(l-1)}{2}+7$. In both cases a square unitizer is found as for the above $2S$, and the other three cases are analogous. $\square$

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