RINGS OF SMALL CLEAN INDEX

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1. Introduction

Definition. An element a of a (unital) ring R is called *clean* if it is a sum of an idempotent and a unit. For a positive integer n, we say that a has *clean index* n if there are precisely n possible clean decompositions of a. For example, a is called *uniquely clean* if it has clean index 1 and a ring R is uniquely clean if all elements have clean index 1.

We use the following notations: $Id(R)$, for the set of idempotents of R, $U(R)$, for the set of units of R and $cn(R)$, for the set of clean elements of R. This way $cn(R) = Id(R) + U(R)$.

Exercise 1. *(i) Show that in any ring the identity* 1 *is the only idempotent unit.*
(ii) The trivial idempotents (i.e., 0 *and* 1*) are uniquely clean in any ring.*

Solution. (i) If $e^2 = e$ is a unit, multiplying by e^{-1} gives $e = 1$.

(ii) Suppose $0 = e + u$ is a clean decomposition of 0 (i.e., $e \in Id(R)$ and $u \in U(R)$). Then $e = -u \in Id(R) \cap U(R)$, so by (i), $e = 1$ and $0 = 1 + (-1)$ is the only clean decomposition of 0. Next, let $1 = e + u$ be a clean decomposition of 1. Then $1 - e \in Id(R) \cap U(R)$, so by (i), $1 - e = 1$, so $e = 0$ and $1 = 0 + 1$ is the only clean decomposition of 1.

Definition. For a positive integer $n \geq 2$, a *clean* ring R is said to have *clean index* n if all elements, excepting 0 and 1 have clean index n.

Notice that the clean index of a ring is less or equal to the number of idempotents of R.

A *more general* definition can be given: an *arbitrary* ring R has clean index n, if all clean elements of R have clean index n.

In this sense, rings of clean index 1, 2 or 3 were described in [1], while rings of clean index 4 were described in [2].

2. Rings of clean index 1

Clearly, a clean ring has clean index 1 iff it is uniquely clean.

A ring is called *Abelian* if it has only central idempotents. For a ring R, the elements included in $U(R) + U(R)$ are calle *2-good* (i.e., sums of two units).

From [1].

Theorem 2. *A ring has clean index 1 iff it is Abelian and all its nonzero idempotent are not 2-good.*

Special characterizations are given for semipotent or potent or neat rings.

3. Rings of clean index 2

Definition. A ring is called *connected* if it has only the trivial idempotents.
It is easy to characterize the connected clean rings of index 2.

Exercise 3. *A connected clean ring has clean index 2 iff it is a division ring.*

Solution. Having to our disposal only two idempotents, for every $a \in R$, $0 \neq a \neq 1$, $a = 0 + a = 1 + (a - 1)$ are the only clean decompositions iff a is a unit (and so is $a - 1$). Hence, excepting 0, the ring has only units, so is a division ring.

A connected ring was called *elemental* if 1 is 2-good.
Generally, from [1], it is proved

Theorem 4. *A ring R has clean index 2 iff one of the following holds:*
(1) R is an elemental ring.
(2) $R = A \times B$ where A is an elemental ring and B has clean index 1.
(3) $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, where A and B have clean index 1 and ${}_AM_B$ is a bimodule with $|M| = 2$.

In particular

Corollary 5. *A ring R is a clean ring with index 2 iff one of the following cases occurs:*
(1) R is a local ring with $R/J(R) \ncong \mathbb{Z}_2$.
(2) $R = A \times B$ where A is a local ring with $A/J(A) \ncong \mathbb{Z}_2$ and B is a uniquely clean ring.
(3) $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, where A, B are uniquely clean rings and ${}_AM_B$ is a bimodule with $|M| = 2$.

4. Rings of clean index 3

From [1]

Theorem 6. *A ring has clean index 3 iff $R = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}$, where A and B have clean index 1 and ${}_AM_B$ is a bimodule with $|M| = 3$.*

Among other results, it is noteworthy to mention that

Theorem 7. *(i) A potent ring of finite clean index is clean.*
(ii) Semipotent or clean or neat rings cannot have clean index 3.

5. Rings of clean index 4

From [2]: "In this article, we characterize the rings of clean index 4. As applications, *clean* rings of clean indexes 4, 5, 6 and 7 are completely determined."

References

[1] T. K. Lee and Y. Zhou *Clean index of rings*. Comm. in Algebra **40** (3) (2012), 807-822.
[2] T. K. Lee and Y. Zhou *Rings of clean index 4 and applications*. Comm. in Algebra **41** (2013), 238-259.