IDEMPOTENTS NOT IDEMPOTENT REVERSIBLE

GRIGORE CĂLUGĂREANU, HORIA F. POP

ABSTRACT. We provide a 3×3 matrix example of two nontrivial idempotents e, f whose product ef is nonzero idempotent but fe is not idempotent.

1. Introduction

The starting point of this note is the following

Proposition 1. *Let R be a commutative domain and $A, B \in \mathbb{M}_2(R)$. If AB is a nonzero idempotent, so is BA.*

Proof. If $AB = I_2$ then also $BA = I_2$, as $\mathbb{M}_2(R)$ is Dedekind-finite. In the remaining case, assume AB is a nontrivial idempotent. By hypothesis on R, $\det(AB) = 0$ and $Tr(AB) = 1$. If $\det(A)\det(B) = 0$, at least one determinant is zero, whence $\det(BA) = 0$. Finally, as $Tr(BA) = Tr(AB)$, it follows that BA is also a nontrivial idempotent. □

Remarks. 1) The case $AB = 0_2$ must be excepted in the previous statement. Indeed, for example, $E_{12}E_{11} = 0_2 \neq E_{12} = E_{11}E_{12}$.

2) What if we start with two idempotent 2×2 matrices E, F whose product EF is also idempotent ? Is FE also idempotent ?

The answer is obviously affirmative if at least one idempotent is trivial. If both are nontrivial, the above proposition applies, so the answer is affirmative, in all cases.

3) Recall **Proposition (21.2)** from [2]: two idempotents e, f are called *isomorphic* iff there exist $a, b \in R$ such that $e = ab$ and $f = ba$.

The goal of this note is to find two nonzero idempotents e, f is some ring R, such that ef is a nonzero idempotent but fe is not.

As observed above, the ring of all 2×2 matrices over any commutative domain won't do.

Again, if we don't except $EF = 0_2$ then an example can be easily found: take $E = E_{22}$, $F = E_{11} + E_{12}$ Then $EF = 0_2$ but $FE = E_{12}$ is not idempotent.

A *GCD* domain is a commutative domain where greatest common divisors exist.

2. The example

A natural choice for finding the example is $\mathbb{M}_3(R)$, for some unital ring R.

As in the 2×2 case, the trivial idempotent matrices are to be excepted.

Recall from [1] the following characterization which can be used when checking a 3×3 matrix is idempotent. However, below a direct computation is also easy.

Theorem 2. *A 3×3 matrix E over an GCD domain R is nontrivial idempotent iff* $\det(E) = 0$, *$rank(E) = Tr(E) = 1 + \frac{1}{2}(Tr^2(E) - Tr(E^2))$ and $rank(E) + rank(I_3 - E) = 3$.*

Computer search for $R = \mathbb{Z}$. Very large number of examples: more than 400 with entries only 0, ± 1.

Example. Over any unital ring, for $E = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $F = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$, the product $EF = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is an idempotent, but $FE = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is not.

References

[1] G. Călugăreanu 3×3 *idempotent matrices over some domains and a conjecture on nil-clean matrices*. Scientific Annals of "Al. I. Cuza" University, **68** (1) (2022), 91-106.

[2] T. Y. Lam *A first course in noncommutative rings*. Second edition. Graduate texts in Mathematics, 131, Springer 2001.