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Rickart matrices over commutative rings

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ABSTRACT. An element of a unital ring is termed right Rickart if its right annihilator is a principal right ideal generated by an idempotent. This paper investigates the properties of right Rickart elements in general rings, with particular emphasis on their behavior in matrix rings over various classes of commutative rings where idempotents are diagonalizable.

1. INTRODUCTION

The rings we consider are supposed to be associative with identity.

Let R be a ring, and let a, e be elements of R. The right annihilator of a, denoted $r(a) = \{x \in R : ax = 0\}$, is defined as the set of elements of R that are annihilated by a on the right. This set forms a right ideal in R. Similarly, eR denotes the right (principal) ideal in R generated by e.

The concept of Rickart rings originates from Rickart's work on C*-algebras, elaborated in the seminal paper [7]. In 1960, it was Maeda [5] who defined a ring as right (or left) Rickart if the right (or left) annihilator of any element is generated by an idempotent. In this note, we provide an elementwise formulation of this concept.

Definition 1.1. Let a be an element of a ring R, and let e be an idempotent in R. The pair (a, e) is called a *right Rickart pair* in R if $r(a) = eR$. The element a is called right Rickart if there exists an idempotent e such that (a, e) is a right Rickart pair. If the idempotent e is unique, a is said to be *uniquely* right Rickart.

The corresponding concepts for left Rickart pairs and left Rickart elements are defined analogously.

For convenience, this note primarily focuses on right Rickart pairs and elements, omitting the term "right" for brevity.

Elements for which $r(a) = 0$ or $r(a) = R$ (associated with trivial idempotents) are termed *trivially* Rickart. In a unital ring, these elements include the left non zero-divisors (in particular, the units) and 0. Thus, every element of any domain are both right and left Rickart. Furthermore, a *connected* ring (i.e., one with only trivial idempotents) is right Rickart if and only if it is a domain.

For $a \in R$, the right ideal aR is a direct summand if and only if a is (von Neumann) regular. Consequently, regular elements - including idempotents and unit-regular elements - are both right and left Rickart.

The primary goal of this note is to characterize the Rickart matrices in $\mathbb{M}_n(R)$, the ring of $n \times n$ matrices over some commutative rings R, for any positive integer n.

In Section 2, we explore general properties of Rickart elements in arbitrary rings. Sections 3 focuses on characterizing Rickart matrices up to similarity over various classes of

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rings. The characterization of Rickart matrices relies heavily on the description of idempotent matrices, making the diagonalization of idempotent matrices particularly significant.

Recall that, following Steger [9], a ring R is called an *ID* ring if every idempotent matrix over R is similar to a diagonal one. Examples of ID rings include division rings, local rings, projective-free rings, principal ideal domains, elementary divisor rings, unit-regular rings, and serial rings.

Steger's results were later generalized by Song and Guo [8], who proved (see Theorem 2) that equivalent idempotents are conjugate.

We denote by E_{ij}, the $n \times n$ matrix with all entries zero except for the (i, j)-entry, which is 1, and by $U(R)$, the set of all units of a ring R.

2. Rickart elements

First, note that in order to check that (a, e) is a Rickart pair, one has to check both inclusions, that is

R1 $r(a) \subseteq eR$, or equivalently, if $ab = 0$ for some $b \in R$ then there is $c \in R$ such that $b = ec$, and

R2 $eR \subseteq r(a)$, or equivalently, $aeb = 0$ for every $b \in R$. Moreover, since the ring is unital, equivalently, $ae = 0$, or equivalently, $e \in r(a)$.

Proposition 2.1. *An element a is regular iff there is an $x \in R$ such that $(a, 1 - ax)$ is a Rickart pair.*

Proof. Suppose a is regular in R and $a = axa$ for some x. We take $e = 1 - xa$, the complementary idempotent to xa. Then, $a(1 - xa) = (a - axa) = 0$, shows that $(1 - xa)R \subseteq r(a)$ (**R2**) and $y = (1 - xa)y$ if $ay = 0$, shows the opposite inclusion (**R1**). Conversely, $r(a) = (1 - xa)R$ implies $a(1 - xa) = 0$ and so $a = axa$ is regular. □

In particular, *idempotents* and *unit-regular* elements are right (and left) Rickart.

It is easy to adapt some well-known proofs (see [4], **7.48**) given for rings, to elements.

Proposition 2.2. *An element $a \in R$ is right Rickart iff aR is projective (as a right R-module).*

Proof. Consider the exact sequence $0 \to r(a) \to R \xrightarrow{f} aR \to 0$, where $f(x) = ax$ for any $x \in R$. Since a is right Rickart, $r(a) = eR$ for some idempotent $e \in R$, so the above sequence splits. This implies that aR is projective. Conversely, if aR is projective, the sequence splits, so $r(a) = eR$ for some idempotent $e \in R$. □

A simple characterization of the right Rickart elements in Abelian rings was proved by Endo (see [4], **Ex. 7.20**).

Proposition 2.3. *If R is Abelian, aR is projective iff $a = be$ with $e = e^2$ and $r(b) = 0$.*

In particular, *in Abelian rings, nonzero nilpotents are not Rickart*.

However, as matrix rings are not Abelian, we cannot use this simple form (a product of an idempotent and a left non-zero-divisor) when describing Rickart matrices.

We proceed with two examples of nonzero nilpotents which are right (or left) Rickart in some not Abelian rings.

Example 2.1. Any nonzero nilpotent in any matrix ring over any right semihereditary ring. Indeed, recall that *a ring R is right semihereditary iff, for every $n \geq 1$, $\mathbb{M}_n(R)$ is right Rickart* (see [4], **7.63**). In particular, any nilpotent matrix of $\mathbb{M}_n(\mathbb{Z})$ is right (and left) Rickart.

Example 2.2. Take $R = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$, a ring which has 5 idempotents 0_2, E_{11}, E_{22}, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, two units I_2, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and a nilpotent $T = E_{21}$. The nilpotent T *is right and left Rickart*.

Indeed, both $r(T) = \{0_2, E_{22}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, T\} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} R$, $l(T) = R \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ are principal ideals generated by idempotents.

This is superseded by a result mentioned in Section 3 (see Proposition 3.8): as $\mathbb{Z}_2$ is a field, $\mathbb{T}_2(\mathbb{Z}_2)$ is Rickart, so all its elements are Rickart (incl. T).

A useful result for the study of Rickart matrices follows.

Proposition 2.4. *(i) If (a, e) is a Rickart pair and $u \in U(R)$ then $(u^{-1}au, u^{-1}eu)$ is also a Rickart pair.*

(ii) Rickart elements are invariant to conjugations.

(iii) Rickart elements are invariant to equivalences.

Proof. (i) (**R1**) Assume $u^{-1}aub = 0$. Then $aub = 0$ and so there exists $c \in R$ such that $ub = ec$. Hence $b = (u^{-1}eu)(u^{-1}c)$.

(**R2**) Indeed, $(u^{-1}au)(u^{-1}eu) = u^{-1}(\underline{ae}u) = 0$.

(ii) Follows from (i).

(iii) Suppose (a, e) is a Rickart pair and $u \in U(R)$. First, as $r(ua) = r(a)$ it follows that (ua, e) is also Rickart. Secondly, as $r(au) = r(u^{-1}au)$, using (i), $(au, u^{-1}eu)$ is a Rickart pair. $\square$

Denote $Ric(R)$ the set of all the Rickart elements of R. As one might expect, owing to the behavior of $r(a)$ and eR, examples show that $Ric(R)$ is **not** closed under addition, nor under multiplication.

As for *addition*, in section 3, it is proved that $2I_2 = I_2 + I_2$ is not Rickart in $\mathbb{M}_2(\mathbb{Z}_4)$. However, I_2 is (trivially) Rickart, being a unit.

As for *multiplication*, recall that idempotents (and more general (unit-)regular elements) are Rickart. Below we provide a product of two idempotents which is not Rickart. Since (see [4], **Ex. 7.10**) a commutative semihereditary ring must be reduced, $\mathbb{Z}_4$ is not semihereditary and it makes sense to search for examples in $\mathbb{M}_2(\mathbb{Z}_4)$, which, as already mentioned, is not (right) Rickart.

Consider the idempotents E_{11} and $E = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}$ in $\mathbb{M}_2(\mathbb{Z}_4)$. The product $E_{11}E = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ is nilpotent (as $2^2 = 0$) in $\mathbb{Z}_4$.

Proposition 2.5. *Let $R := \mathbb{M}_2(\mathbb{Z}_4)$. Then $A := \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ is not Rickart.*

Proof. Otherwise, $r_R(A) = FR$ for some idempotent matrix $F \in R$. Since we have $2E_{11}, 2E_{22} \in r_R(A) = FR$, this implies $(I_2 - F)2E_{11} = 0 = (I_2 - F)2E_{22}$. Write

$$I_2 - F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then

$$0 = (I_2 - F)2E_{11} = \begin{bmatrix} 2a & 0 \\ 2c & 0 \end{bmatrix} \text{ and } 0 = (I_2 - F)2E_{22} = \begin{bmatrix} 0 & 2b \\ 0 & 2d \end{bmatrix}$$

Hence

$$I_2 - F = \begin{bmatrix} 2a' & 2b' \\ 2c' & 2d' \end{bmatrix},$$

where $a', b', c', d' \in \mathbb{Z}_4$. Then $A(I_2 - F) = 0$ and $AF = 0$, implying $A = 0$, a contradiction. □

Remark. While $U(R)e \subseteq Ric(R)$, as these are unit-regular elements, Re is generally not included in $Ric(R)$: an example is $(2I_2)(E_{11} + E_{12}) = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_4$, as witnessed by the previous proposition.

Similarly, for $n \times n$ matrices, taking $A = E_{11}E$, with $E := \sum_{j=1}^{n}(2E_{1j} + 3E_{2j})$, we have the following.

Proposition 2.6. *Let $R := \mathbb{M}_n(\mathbb{Z}_4)$, $n > 1$. Then $A := \sum_{j=1}^{n} 2E_{1j}$ is not Rickart.*

In what follows, we characterize the uniquely Rickart elements. Here, for some subset X of R, $\ell_R(X)$ denotes the left annihilator of X in R.

Theorem 2.1. *An element $a \in R$ is uniquely Rickart iff $r_R(a)\ell_R(r_R(a)) = 0$.*

Proof. To show the condition is necessary, write $r_R(a) = eR$, where $e = e^2 \in R$. Then $eR = (e + ex(1-e))R$ for any $x \in R$ and so $ex(1-e) = 0$. That is, $eR(1-e) = 0$ and so $r_R(a)\ell_R(r_R(a)) = 0$ as $R(1-e) = \ell_R(r_R(a))$.

The condition is also sufficient. Let $r_R(a) = eR = fR$, where e, f are idempotents of R. Then $ef = f$ and $fe = e$ and so $(1-f)eR = 0$. Hence $1 - f \in \ell_R(r_R(a))$ and since $r_R(a)\ell_R(r_R(a)) = 0$, we get $eR(1-f) = 0$ and so $e(1-f) = 0$. Thus $e = ef = f$, as desired. □

Corollary 2.1. *An element $a \in R$ is uniquely Rickart iff there exists an idempotent $e \in R$ such that $r_R(a) = eR$ and $eR(1-e) = 0$.*

As it is known, an idempotent is called *right semicentral* if $eR(1-e) = 0$. Hence a Rickart pair (a, e) is uniquely Rickart iff e is right semicentral.

Corollary 2.2. *A nonzero element a in a prime ring R is uniquely Rickart iff $r_R(a) = 0$.*

As already done above with uniquely Rickart elements, following the pattern of clean or nil-clean or fine elements, we can consider the following subclass of Rickart elements.

Definition 2.2. An *element* a in a ring R is called *strongly Rickart* if there exists an idempotent $e \in R$ such that (a, e) is a Rickart pair and $ae = ea$. This means that not only $ae = 0$ (by **(R2)**) but also $ea = 0$.

Strongly Rickart elements can be also characterized by an apparently weaker condition.

Proposition 2.7. *In a ring R, an element $a \in R$ is strongly Rickart iff there exists an idempotent $e \in R$ such that (a, e) is a Rickart pair and $ea \in Ra^2$.*

Proof. Since the "only if" part is trivial as $ae = ea = 0$, assume that there exists an idempotent $e \in R$ such that (a, e) is a Rickart pair and $ea \in Ra^2$. Then $ea = ra^2$ for some $r \in R$. In particular, $ea = era^2$. Therefore,

$$(e - era(1-e))a = 0.$$

Let $f := e - era(1-e)$. Then f is an idempotent of R such that $r_R(a) = eR = fR$ and $fa = 0$. From $ae = 0$ we also get $af = 0$, as desired. □

Corollary 2.3. *In a ring R, if a is Rickart and $a \in Ra^2$, then a is strongly Rickart.*

The pair (a, e) for an idempotent e is a *left Rickart* in R if $\ell_R(a) = Re$.

The strongly Rickart elements may not be left Rickart elements. Indeed, take a a left non-zero-divisor which is a right zero-divisor. Then $r(a) = 0$, $a \cdot 0 = 0 \cdot a = 0$ but $\ell_R(a) \neq 0$.

Note that a notion of strongly Rickart ring also appears in the literature - namely, a ring in which the right annihilator of each single element of R is generated by a left semicentral idempotent (see [1]). This differs from our elementwise formulation above. In the terminology we use here, [1] calls a ring strongly Rickart if every element $a \in R$ occurs in a Rickart pair (a, e) where the idempotent e is left semicentral; that is, $ere = er$ for all $r \in R$. In our work we do not consider strongly Rickart rings in this global sense—our study is strictly elementwise.

In order to simplify the wording and the notations, in closing this section, we introduce here two subsets related to Rickart pairs: for a Rickart element a, $\mathcal{E}(a) := \{e \in R : e = e^2, r(a) = eR\}$, and, for an idempotent $e \in R$, $Ric(e) := \{a \in R : r(a) = eR\}$.

Clearly, for any Rickart element a, $\mathcal{E}(a)$ is not empty and a was termed uniquely Rickart if $\mathcal{E}(a)$ consists of a single element. As already proved in Corollary 2.1, this holds iff the idempotent is right semicentral. As it contains at least $1 - e$, $Ric(e)$ is also not empty.

Next, we give an example of Rickart element a such that $\mathcal{E}(a)$ consists of 4 idempotents.

Example 2.3. The nilpotent $E_{11}(E_{12} + E_{21}) = E_{12}$, as an idempotent by unit product, is unit-regular, so Rickart. We consider this product over $\mathbb{Z}_4$ and we show that $\mathcal{E}(E_{12})$ consists of 4 idempotents.

First note that $r(E_{12}) = \{aE_{21} + bE_{22} : a, b \in \mathbb{Z}_4\}$. Such a linear combination is an idempotent iff $a = ab$ and $b = b^2$. As $\mathbb{Z}_4$ has only trivial idempotents, if $b = 0$ then $a = 0$ and clearly 0_2 does not generate $r(E_{12})$. If $b = 1$, we have four idempotents $E_{22} + aE_{21} = \begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$, for each $a \in \mathbb{Z}_4$. Each of these generates $r(E_{12})$: indeed, $\begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ z & w \end{bmatrix}$ covers all $r(E_{12})$. It is easy to check that the idempotent matrices $\begin{bmatrix} 0 & 0 \\ a & 1 \end{bmatrix}$ are not right semicentral.

As already mentioned, elements such that $r(a) = 0$ or $r(a) = R$ are called trivially (right) Rickart. These are the left non-zero-divisors and 0, respectively.

Rephrasing, for any left non-zero-divisor $z \in R$, $\mathcal{E}(z) = 0$ and $Ric(1) = 0$. Hence, all left non-zero-divisors (in particular, the units) are uniquely Rickart.

As already mentioned, these may not be left Rickart.

The nontrivial idempotents are Rickart but (in general) not uniquely Rickart. Actually, for any nontrivial idempotent e, $E(e)$ consists (at least) of the complementary idempotent $1 - e$ (as $r(e) = (1 - e)R$ for any idempotent). However, if e and f are idempotents and $f \in \mathcal{E}(e)$, it does not follow $f = 1 - e$.

Since $1 - e$ is right semicentral iff e is left semicentral, according to Corollary 2.1, for an example it suffices to start with an idempotent which is not left semicentral.

Example 2.4. For $e = E_{11}$, $f = E_{21} + E_{22}$ we have $ef = 0$. If $eB = E_{11} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 0_2$ then $B = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$. As $B = fC = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x + z & y + t \end{bmatrix}$, it suffices to choose $x + z = c$, $y + t = d$. For example, $C = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$ suits well, but there are also many other choices. Clearly, $f \neq 1 - e = E_{22}$.

Actually, it is readily checked that the idempotent E_{11} is not left semicentral.

3. Rickart matrices over some commutative domains

We first recall some known results on matrix rings over semi-hereditary rings.

A ring R is *right (semi-)hereditary* if all (finitely generated) right ideals of R are projective modules. Semisimple rings are left and right hereditary, von Neumann regular rings are left and right semihereditary. The right Bézout domains are right semihereditary, PIDs are right and left hereditary.

Of utmost importance (see [4], **7.63**) for our paper is the following result.

Theorem 3.2. *A ring R is right semihereditary iff, for every integer $n \geq 1$, the matrix ring $S = \mathbb{M}_n(R)$ is right Rickart.*

Consequently, matrix rings over Bézout domains are right Rickart.

As for the upper triangular $n \times n$ matrices, we recall (see [4], **Ex. 7.25**)

Proposition 3.8. *For a domain D and a fixed integer $n > 1$, $\mathbb{T}_n(D)$ is a right Rickart ring iff D is a division ring.*

As a nonexample of the above proposition, we mention $2I_2 \in \mathbb{T}_2(\mathbb{Z}_4)$. It is readily checked that $r(2I_2) = 2\mathbb{M}_2(\mathbb{Z}_4)$ contains (only zero-square matrices and so) no nonzero idempotent. Therefore $2I_2$ is not (right) Rickart.

We also recall two results from [2] (see **4.11** and **5.3**).

Theorem 3.3. *(i) Let $A \in \mathbb{M}_n(R)$, for a commutative ring R. Then $\operatorname{rank}(A) < n$ iff $\det(A)$ is a zero divisor.*

(ii) (N. McCoy) Let $A \in \mathbb{M}_{m\times n}(R)$, for a commutative ring R. The homogeneous system of equations $AX = 0$ has a nontrivial solution iff $\operatorname{rank}(A) < n$ (the number of unknowns),

3.1. Rickart matrices paired with some block diagonal idempotents.

The goal of this subsection is to describe the Rickart pairs in $\mathcal{E}(\operatorname{diag}(I_k, E))$ for some $(n-k) \times (n-k)$ idempotent submatrix E and $1 \leq k \leq n-1$.

We first deal with the 2×2 case.

Theorem 3.4. *Over any domain R, the nontrivial Rickart 2×2 matrices which are paired with $\operatorname{diag}(1, e)$, for some idempotent $e \in R$, are the nonzero matrices $A \in \mathbb{M}_2(R)$ with zero first column and $\operatorname{col}_2(A)e = \mathbf{0}$.*

Proof. We just check the inclusions (**R2**) and (**R1**) for $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$.

(**R2**) $A\operatorname{diag}(1, e) = 0_2$ gives $a_{11} = a_{21} = 0$ and $a_{12}e = a_{22}e = 0$.

(**R1**) If $AB = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = 0_2$ then, as R is a domain (and a_{12}, a_{22} are not both zero), we get $b_{21} = b_{22} = 0$. Then, for the factorization $B = \operatorname{diag}(1, e)C$, we can choose $C = B$. □

Remark. If $e = 0$, these are the Rickart matrices paired with E_{11}.

Next, we address the general case, that is, we determine the Rickart matrices which belong to $\mathcal{E}(\operatorname{diag}(I_k, E))$ for some $(n-k) \times (n-k)$ idempotent submatrix E.

Theorem 3.5. *Let R be a commutative domain, $1 \leq k \leq n-1$ and E an $(n-k) \times (n-k)$ idempotent matrix. For a matrix $A \in \mathbb{M}_n(R)$, denote by $A_{n,n-k}$ the $n \times (n-k)$ submatrix of A obtained by removing the first k columns. Then the pair $(A, \operatorname{diag}(I_k, E))$ is a nontrivial Rickart pair iff the following conditions are satisfied:*

(a) the first k columns of A are zero,

(b) $A_{n,n-k}E = 0_{n\times(n-k)}$ and

(c) for an $n \times n$ matrix B, if $AB = 0_n$ then $rank(E) = rank(E, \mathrm{col}_j^{(k+1)}(B))$ for each $1 \le j \le n$.

Proof. We start with (**R2**), that is, $A\mathrm{diag}(I_k, E) = 0_n$. An easy computation shows this holds precisely when the first k columns of A are zero and $A_{n,n-k}E = 0$. Notice that if $rank(A_{n,n-k}) = n-k$ then by Theorem 3.3, (ii), $E = 0_{n-k}$.

As for (**R1**), suppose

$$AB = \begin{bmatrix} 0 & \cdots & 0 & a_{1,k+1} & \cdots & a_{1,n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{n,k+1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = 0_n,$$ for some B. This is equivalent to some homogeneous linear systems of n equations and $n-k$ unknowns. The first one refers to the lower $n-k$ entries in the first column of B, namely

$$\begin{cases} a_{1,k+1}b_{k+1,1} & + & \cdots & + & a_{1n}b_{n1} & = & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ a_{n,k+1}b_{k+1,1} & + & \cdots & + & a_{nn}b_{n1} & = & 0 \end{cases}$$

and the others, to the lower $n-k$ entries of the second column of B, and so on, ending with the lower $n-k$ entries in the n-th column of B.

Denote by $\mathrm{col}_j^{(k+1)}(B) = \begin{bmatrix} b_{k+1,j} \\ b_{k+2,j} \\ \vdots \\ b_{nj} \end{bmatrix}$, for $1 \le j \le n$, that is, the lower part of $\mathrm{col}_j(B)$.

The above equations can be written $A_{n,n-k}\mathrm{col}_j^{(k+1)}(B) = 0_{n\times 1}$ for every $1 \le j \le n$.

We continue with the second part of (**R1**). The product

$$\mathrm{diag}(I_k, E)C = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & e_{11} & \cdots & e_{1,n-k} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & e_{n-k,1} & \cdots & e_{n-k,n-k} \end{bmatrix} C$$

has the first k rows of C as first k rows. Hence, if a matrix C exists with $B = \mathrm{diag}(I_k, E)C$, the first k rows of $C = [c_{ij}]$ must be the first k rows of B.

Further, the last $n-k$ rows of C are solutions of some (possibly nonhomogeneous) linear systems. The linear systems are $E\mathrm{col}_j^{(k+1)}(C) = \mathrm{col}_j^{(k+1)}(B)$ with $1 \le j \le n$, with the entries of the last $n-k$ rows of C as unknowns. Each system has E as system matrix and the RHS of the equations are entries of B which satisfy $AB = 0_n$. Denoting $B_{n-k,n}$ the submatrix of B which consists of the last $n-k$ rows, this amounts to $A_{n,n-k}B_{n-k,n} = 0_n$.

By Kronecker-Rouché-Capelli theorem, the linear systems above (which give the last $n-k$ rows of C) are solvable iff $rank(E) = rank(E, \mathrm{col}_j^{(k+1)}(B))$ for each $1 \le j \le n$. □

Remark. If $rank(A) = rank(A_{n,n-k}) = n-k$ then $E = 0_{n-k}$, the last $n-k$ rows of B are zero and we can choose $C = B$.

3.2. **Rickart matrices, up to similarity, over some commutative domains.** In this subsection, up to similarity, we apply the results obtained in the previous subsection.

3.2.1. *Over GCD domains.* First recall that a *commutative* domain is a *GCD* domain if every pair a, b of nonzero elements has a greatest common divisor, denoted by $\gcd(a, b)$. GCD domains include unique factorization domains, Bézout domains and valuation domains. A basic property of a GCD domain is needed for the next proposition: if a divides bc and $\gcd(a, b) = 1$ in a GCD domain, then a divides c.

In the 2×2 case, the following result is known. For reader's convenience, we supply a proof.

Proposition 3.9. *Let R be a GCD domain. Then every nontrivial idempotent 2×2 matrix is similar to E_{11}.*

Proof. Over any integral domain it is well-known that every nontrivial idempotent 2×2 matrix has the form $E = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ with $bc = a(1-a)$ (i.e., zero determinant and trace $= 1$). Let $x = \gcd(a, c)$ and $a = xy$, $c = xx'$ with $\gcd(y, x') = 1$. Since $bx' = y(1-a)$, by the GCD hypothesis, y divides b, say $b = yy'$. Now take $P = \begin{bmatrix} x & y' \\ -x' & y \end{bmatrix}$. One can check that $\det(P) = 1$ (if $a = 0$ then $b = 0$ or $c = 0$ and $P = \begin{bmatrix} 0 & 1 \\ -1 & b \end{bmatrix}$ and $P = \begin{bmatrix} c & 1 \\ 1 & 0 \end{bmatrix}$, respectively; if $a \neq 0$, one checks $a \det(P) = a$) and $PE = E_{11}P$. Hence E is similar to E_{11}. □

Essential in the proof of Proposition 3.9, is that we know the precise form of the nontrivial idempotent 2×2 matrices over commutative domains.

As for the 3×3 case, the situation is harder, since the characterization of the idempotents is as follows (see [3]): A 3×3 matrix E over a GCD domain R is a nontrivial idempotent iff $\det(E) = 0$, $rank(E) = \mathrm{Tr}(E) = 1 + \frac{1}{2}(\mathrm{Tr}^2(E) - \mathrm{Tr}(E^2))$ and $rank(E) + rank(I_3 - E) = 3$.

However, we succeeded proving an analogous result in the 3×3 case.

Proposition 3.10. *Let R be a GCD domain. Then every nontrivial idempotent 3×3 matrix is similar to E_{11} or $E_{11} + E_{22}$.*

Proof. First notice that it suffices to deal with the *nontrivial rank one* idempotent case, that is, if E is a rank one (and so trace one) nontrivial idempotent 3×3 matrix over a commutative (GCD) domain and there exists an invertible 3×3 matrix U such that $EU = UE_{11}$, then it follows that $(I_3 - E)U = U(I_3 - E_{11}) = U(E_{22} + E_{33})$, and $E_{11} + E_{22}$ is conjugate with $E_{22} + E_{33}$ by $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

Secondly, from [3] (the description of cases is somewhat analogous in [6]), any rank one (and trace one) 3×3 (nontrivial) idempotent matrix has the form

$$E_1 = \begin{bmatrix} 1 - sa_2 - va_3 & s(1 - sa_2 - va_3) & v(1 - sa_2 - va_3) \\ a_2 & sa_2 & va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0 & a_1 & sa_1 \\ 0 & 1 - sa_3 & s(1 - sa_3) \\ 0 & a_3 & sa_3 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & b \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, in each case we provide a suitable unit.

For E_3 it is easy to find a unit $U = [u_{ij}]$ such that $E_3U = UE_{11}$. The only conditions are $u_{32} = u_{33} = 0$, $u_{11} = au_{31}$ and $u_{21} = bu_{31}$. Hence, $U = \begin{bmatrix} a & u_{12} & u_{13} \\ b & u_{22} & u_{23} \\ 1 & 0 & 0 \end{bmatrix}$ where we just choose the upper right 2×2 minor for U to be a unit. For example, $U = \begin{bmatrix} a & 1 & 0 \\ b & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ suits well. It is readily verified that $E_3U = UE_{11} = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, *over any commutative ring*.

For E_2 (we assume $s, a_1, a_3 \neq 0$), by computation we get $u_{22} + su_{32} = 0 = u_{23} + su_{33}$ and $u_{11} = a_1(u_{21} + su_{31})$, $u_{21} = (1 - sa_3)(u_{21} + su_{31})$, $u_{31} = a_3(u_{21} + su_{31})$.

This requires $a_1 \mid u_{11}$, $(1 - sa_3) \mid u_{21}$ and $a_3 \mid u_{31}$. The simplest choice is to take equalities, i.e., $a_1 = u_{11}$, $1 - sa_3 = u_{21}$ and $a_3 = u_{31}$. Then we also need $u_{21} + su_{31} = 1$, which indeed holds, and so $U = \begin{bmatrix} a_1 & u_{12} & u_{13} \\ 1 - sa_3 & -su_{32} & -su_{33} \\ a_3 & u_{32} & u_{33} \end{bmatrix}$. We need a choice of the u_{ij}'s, in order to complete the first column to an invertible matrix.

Observe that $\det(U) = a_1 \cdot 0 - (1 - sa_3) \det \begin{bmatrix} u_{12} & u_{13} \\ u_{32} & u_{33} \end{bmatrix} - sa_3 \det \begin{bmatrix} u_{12} & u_{13} \\ u_{32} & u_{33} \end{bmatrix} = - \det \begin{bmatrix} u_{12} & u_{13} \\ u_{32} & u_{33} \end{bmatrix}$. Thus a good choice is I_2 for this 2×2 minor of U.

Finally $U = \begin{bmatrix} a_1 & 1 & 0 \\ 1 - sa_3 & 0 & -s \\ a_3 & 0 & 1 \end{bmatrix}$, and indeed one can verify $E_2U = UE_{11} = \begin{bmatrix} a_1 & 0 & 0 \\ 1 - sa_3 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix}$.

For E_1, denote $\sigma := 1 - sa_2 - va_3$ and so $E_1 = \begin{bmatrix} \sigma & s\sigma & v\sigma \\ a_2 & sa_2 & va_2 \\ a_3 & sa_3 & va_3 \end{bmatrix}$. By computation (for $E_1U = UE_{11}$), we get $u_{12} + su_{22} + vu_{32} = 0 = u_{13} + su_{23} + vu_{33}$ and $u_{11} = \sigma(u_{11} + su_{21} + vu_{31})$, $u_{21} = a_2(u_{11} + su_{21} + vu_{31})$, $u_{31} = a_3(u_{11} + su_{21} + vu_{31})$.

We choose $u_{11} = \sigma$, $u_{21} = a_2$, $u_{31} = a_3$ and so indeed $u_{11} + su_{21} + vu_{31} = 1$. So far we have $U = \begin{bmatrix} \sigma & -su_{22} - vu_{32} & -su_{23} - vu_{33} \\ a_2 & u_{22} & u_{23} \\ a_3 & u_{32} & u_{33} \end{bmatrix}$.

Now denote $\delta := \det \begin{bmatrix} u_{22} & u_{23} \\ u_{32} & u_{33} \end{bmatrix}$. Then $\det(U) = \sigma\delta + sa_2\delta + va_3\delta = \delta(\sigma + sa_2 + va_3) = \delta$, so a good choice is now $\begin{bmatrix} u_{22} & u_{23} \\ u_{32} & u_{33} \end{bmatrix} = I_2$.

Finally, $U = \begin{bmatrix} \sigma & -s & -v \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{bmatrix}$ (so $\det(U) = 1$) and indeed one can verify $E_1U = UE_{11} = \begin{bmatrix} \sigma & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{bmatrix}$, which completes the proof. □

Open question. Let R be a GCD (commutative) domain. Is every $n \times n$ idempotent matrix, similar with $\sum_{i=1}^{k} E_{ii}$, for some $1 \leq k \leq n$?

In [3], for $n = 3$, this was proved using a lemma: Let R be a GCD (commutative) domain and let C_1, C_2 be two 3×1 nonzero columns. If C_1, C_2 are linearly dependent over R, there exists a column C and elements $a_1, a_2 \in R$ such that $C_i = a_i C$, $i \in \{1, 2\}$. A repetitive procedure (we get C for C_1, C_2 and we repeat the procedure for C, C_3), takes care of the case with three columns, every two of which are dependent, that is, of nontrivial rank one (and so trace one) idempotent 3×3 matrices.

We were not able to extend this to rank two, and to prove the above statement for higher ranks and dimensions of (square) matrices.

Note that, unless being Bézout, a GCD domain need not be (right) semihereditary, and so the corresponding full matrix ring need not be Rickart. Hence it makes sense to describe (up to similarity) the Rickart matrices over GCD domains.

For the 2×2 case, we first discard from $\mathbb{M}_2(R)$ the zero matrix and the left non-zero-divisors matrices (these being trivially Rickart).

A (remaining) matrix $A \in \mathbb{M}_2(R)$ is Rickart iff there is a *nontrivial* idempotent $E \in \mathbb{M}_2(R)$ such that (A, E) is a Rickart pair. According to Proposition 2.4, for any unit $U \in \mathbb{M}_2(R)$, $(U^{-1}AU, U^{-1}EU)$ is also a Rickart pair.

Over a GCD domain, let $V \in U(\mathbb{M}_2(R))$ be such that $V^{-1}EV = E_{11}$. Then $(V^{-1}AV, E_{11})$ is a Rickart pair and so up to similarity, all the nontrivial Rickart matrices in $\mathbb{M}_2(R)$ belong to $\mathcal{E}(E_{11})$. Using Theorem 3.4 we obtain at once

Theorem 3.6. *Over any GCD domain R, up to similarity, the nontrivial Rickart 2×2 matrices are the nonzero matrices $A \in \mathbb{M}_2(R)$ with zero first column.*

For the 3×3 case, we use Theorem 3.5, $k \in \{1, 2\}$ and $E = 0$, and obtain

Theorem 3.7. *Over any GCD domain R, up to similarity, the nontrivial Rickart 3×3 matrices are the nonzero matrices $A \in \mathbb{M}_2(R)$ with zero first column or zero first and second column.*

3.2.2. *$n \times n$ Rickart matrices over ID, commutative, connected rings.* This class of rings includes: ID domains (in particular, PIDs) and local rings. From definitions, we have the following similarity.

Proposition 3.11. *Let A be an $n \times n$ idempotent matrix over an ID connected ring. Then A is similar to $\operatorname{diag}(I_k, 0_{n-k})$ for some $1 \leq k \leq n$.*

Using this and Theorem 3.5, we can describe, up to similarity, the Rickart $n \times n$ matrices.

Theorem 3.8. *Let R be an ID, commutative and connected ring, $A \in \mathbb{M}_n(R)$ and let $E = \sum_{i=1}^{k} E_{ii}$ for some $1 \leq k < n$. Then (A, E) is a Rickart pair iff the first k columns of A are zero and the $n \times (n-k)$ remaining submatrix of A has rank $n-k$. That is, the submatrix of A formed by the last $n-k$ columns contains a $(n-k) \times (n-k)$ submatrix whose determinant is a non-zero-divisor.*

3.2.3. *$n \times n$ Rickart matrices using Song, Guo diagonalization.* In [8] (see Theorem 2), an important result was proved.

Theorem 3.9. *Let R be a ring, $a, b \in R$ with $a^2 = a$ and $b^2 = b$. Then a and b are equivalent iff a and b are conjugate.*

In this subsection we start with a consequence of the above theorem (see [8], Corollary 6).

Proposition 3.12. *Let A be an $n \times n$ idempotent matrix over a ring R. If A has an invertible $k \times k$ submatrix, $1 \leq k \leq n$, then A is similar to $\operatorname{diag}(I_k, E)$ for some (idempotent submatrix) E.*

Indeed, by elementary transformations, the invertible $k \times k$ submatrix can be put at the left-up corner of A, so A is equivalent to $\operatorname{diag}(I_k, E)$. Then by the theorem above, A is similar to $\operatorname{diag}(I_k, E)$.

In the 2×2 case we have

Corollary 3.4. *Let E be an 2×2 nontrivial idempotent matrix over any ring R. If E has a unit entry, then E is similar to $\operatorname{diag}(1, e)$ for some idempotent $e \in R$.*

Using Theorem 3.4 we now obtain

Theorem 3.10. *Over any commutative domain R, the nontrivial Rickart 2×2 matrices which are paired with $\operatorname{diag}(1, e)$ are the nonzero matrices $A \in \mathbb{M}_2(R)$ with zero first column.*

Analogously, in the general $n \times n$ case, using Theorem 3.5 we can determine the Rickart matrices which belong to $\mathcal{E}(\operatorname{diag}(I_k, E))$ for some $(n-k) \times (n-k)$ idempotent submatrix E, that is, are the matrices A such that $(A, \operatorname{diag}(I_k, E))$ is a Rickart pair.

Theorem 3.11. *Over any commutative domain R, the nontrivial Rickart $n \times n$ matrices which are paired with $\operatorname{diag}(I_k, E)$, for $1 \leq k \leq n-1$ and some idempotent $(n-k) \times (n-k)$ matrix, are the nonzero matrices $A \in \mathbb{M}_n(R)$ which have zero the first k columns and the remaining $n \times (n-k)$ submatrix has rank $n-k$.*

References

[1] S. A. Al-Saadi, T. A. Ibrahiem *Strongly Rickart rings*. Mathematical Theory and Modeling, **4** (8) (2014), 95-104.

[2] W. C. Brown *Matrices over commutative rings*. 1993 Marcel Dekker Inc.

[3] G. Călugăreanu 3×3 *idempotent matrices over some domains and a conjecture on nil-clean matrices*. Scientific Annals of "Al. I. Cuza" University, **68** (1) (2022), 91-106.

[4] T. Y. Lam *Lectures on Modules and Rings*. Graduate texts in mathematics, 189, 1999 Springer Verlag.

[5] S. Maeda *On a ring whose principal right ideals generated by idempotents form a lattice*. J. Sci. Hiroshima Univ. Ser. A, **24** (1960), 509-525.

[6] R. B. Richter, W. P. Wardlow *Diagonalization over commutative rings*. Amer. Math. Monthly, **97** (3) (1990), 223-227.

[7] C. E. Rickart *Banach algebras with an adjoint operation*. Ann. of Math., **47** (1946), 528-550.

[8] G. Song, X. Guo *Diagonability of idempotent matrices over noncommutative rings*. Linear Algebra and its Applications, **297** (1999), 1-7.

[9] A. Steger *Diagonability of idempotent matrices*. Pacific J. Math., **19** (3) (1966), 535-542.

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