OOI RINGS

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ABSTRACT. In Lewallen [3], a nonzero R-module M is called an *OI* ("onto-implies-invertible") module if, for all $x \in R$, surjectivity of the multiplication map ρ_x forces x to be invertible in R. A ring R is called *OI* if all its R-modules are OI.

In this note we introduce and study the dual notion: a nonzero R-module M is said to be *OOI* ("one-to-one-implies-invertible") if injectivity of ρ_x implies that x is a unit of R. A ring R is called OOI if every R-module satisfies this condition.

Our main result shows that the class of OI rings and OOI rings coincide: a ring is OOI if and only if it is local with nil maximal ideal. Thus, at the ring level, the OI and OOI properties are equivalent. In contrast, for $\mathbb{Z}$-modules (i.e., Abelian groups) the OOI condition exhibits behavious markedly different from that of OI modules, and several examples are provided to illustrate this phenomenon.

1. INTRODUCTION

Let R be a commutative ring with identity and let M_R denote a unitary right R-module. For each $x \in R$, we denote by $\rho_x : M_R \to M_R$, $\rho_x(m) = mx$ for all $m \in M$, the right-multiplication map induced by x.

In [3], Lewallen introduced the notion of an *OI* ("onto-implies-invertible") module: a nonzero R-module M is OI if, for every $x \in R$, surjectivity of ρ_x implies that x is invertible in R. A ring R is called *OI* if every R-module is OI. These rings were completely characterized in [3]: using the terminology of [2], a ring R is OI if and only if it is local with nil maximal ideal.

Motivated by this result, it is natural to ask whether an analogous characterization holds when surjectivity is replaced by injectivity. Injectivity of multiplication maps is a fundamental condition in module theory, closely related to torsion phenomena and to classical notions such as coHopfian modules, where every injective endomorphism is an automorphism. This leads us to consider the following dual condition.

A nonzero R-module M is called an *OOI* ("one-to-one-implies-invertible") module if, for every $x \in R$, injectivity of ρ_x implies that x is invertible in R. Clearly, if x is a unit, ρ_x is injective; the OOI condition requires the converse implication. A ring R is said to be *OOI* if every R-module is OOI.

The primary aim of this paper is to determine the class of OOI rings and to compare it with the class of OI rings. Despite the apparent asymmetry between surjectivity and injectivity, our main result shows that no new ring-theoretic behavior

Keywords: OI module, OOI module, localization, local ring, Abelian group. MSC 2020 Classification: 13H99, 16D99, 20K99. Orcid: 0000-0002-3353-6958.

arises: a ring is OOI if and only if it is local with nil maximal ideal. Consequently, the OI and OOI properties coincide at the level of rings.

The proof relies on a localization argument inspired by the use of the quotient ring $R[x]/(1-sx)$ for an element $s \in R$. An observation that significantly simplifies the proof in [3], is the isomorphism $S^{-1}R \cong R[x]/(1-sx)$, where $S^{-1}R$ is the localization. Here s is not nilpotent and the multiplicative set $S = \{1, s, s^2, \dots\}$ is considered.

Finally, we turn to the category of $\mathbb{Z}$-modules, that is, Abelian groups. Here the situation changes dramatically: while OI and OOI coincide for rings, they diverge sharply for groups. We present several examples showing that OOI Abelian groups behave very differently from OI groups, highlighting the limitations of duality between injectivity and surjectivity at the module level.

2. The characterization

First note that ρ_r is injective if and only if the (submodule) annihilator of r in M,

$$Ann_M(r) := \{\, m \in M : mr = 0 \,\},$$

is zero; equivalently, ρ_r is injective if and only if

$$mr = 0 \;\;\Rightarrow\;\; m = 0.$$

Proposition 2.1. *Let R be a local ring such that the maximal ideal is nil. Then R is an OOI ring.*

Proof. Suppose $Ann_M(r) = \{0\}$ for some nonzero R-module M. If $r^n = 0$ in R for some $n \geq 1$, then for any $m \in M$ we have $r^n m = 0$. In particular, $r^{n-1}m \in Ann_M(r) = \{0\}$, so $r^{n-1}m = 0$ for all m. Iterating this argument forces $m = 0$ for all $m \in M$, contradicting the assumption $M \neq 0$. Thus r cannot be nilpotent.

Since R is local, every element is either a unit or nilpotent. Therefore, if ρ_r is injective on M, then r must be invertible. Hence R is OOI. □

As mentioned in the Introduction, for the converse we use a localization construction suggested by the use (in [3]) of the quotient ring $R[x]/(1-sx)$. Since we could not find a (book) reference in the literature, we provide a proof here.

Lemma 2.2. *If $S = \{1, s, s^2, \dots\}$ for some $s \in R$, then*

$$S^{-1}R \;\cong\; R[x]/(1-sx).$$

Proof. Set $A := R[x]/(1-sx)$. Then A is an R-algebra in which the class of s is invertible (its inverse being the class of x). By the universal property of localization, there is a unique R-algebra homomorphism

$$S^{-1}R \longrightarrow A.$$

Conversely, sending $x \mapsto 1/s$ defines an R-algebra map $R[x] \to S^{-1}R$ that vanishes on $1-sx$, hence induces a map

$$A \longrightarrow S^{-1}R.$$

These two maps are inverses of each other, so $A \cong S^{-1}R$.

Under this isomorphism one has

$$\frac{r}{s^n} \;\longmapsto\; rx^n \pmod{(1-sx)},$$

so x corresponds to $1/s$. □

Proposition 2.3. *Let R be an OOI ring. Then R is local with nil maximal ideal.*

Proof. Assume R is OOI and let $r \in R$ be a nonunit. Suppose, for contradiction, that r is not nilpotent. Consider $S = \{1, r, r^2, \dots\}$ and the localization $S^{-1}R$, which is a nonzero R-module because r is not nilpotent.

In $S^{-1}R$ the element $\frac{r}{1}$ is a unit, so multiplication by r on $S^{-1}R$ is invertible, hence injective. By the OOI property, this forces r to be a unit in R, a contradiction. Therefore, every nonunit is nilpotent. Thus R is local with nil maximal ideal. □

Fields are a special case of reduced rings. There exist OOI rings that are not fields; for example, $k[x]/(x^2)$ (for a field k) is OOI because the only nonunit is the nilpotent class of x, yet it is not a field.

Analogously with the above proof, the three-step argument in [3] for the implication "OI ⇒ local with nil maximal ideal" can be streamlined as follows.

Proof. Suppose $x \in R$ is a nonunit and not nilpotent. Consider the multiplicative set $S = \{1, x, x^2, \dots\}$. Then the localization $M = S^{-1}R$ is nonzero. In M, the element x becomes invertible, so multiplication by x is bijective, hence surjective. By the OI property, this forces x to be a unit in R, a contradiction. Thus every nonunit is nilpotent. □

3. Examples of OOI Abelian groups

In this section, the word *group* always means an Abelian group.

A group G is called an *OOI-group* if

$$G[n] := \{\, x \in G : nx = 0 \,\} = 0 \quad \Longrightarrow \quad n \in \{\pm 1\}.$$

While OOI rings coincide with OI rings, the situation for $\mathbb{Z}$-modules (i.e. Abelian groups) is very different: OOI groups behave drastically unlike OI groups (for a comprehensive study of OOI Abelian groups, see [1]). We list a few illustrative examples below.

(1) It is straightforward to see that $\mathbb{Z}$ and every direct sum of copies of $\mathbb{Z}$ (i.e. free groups) are OI groups. However, $\mathbb{Z}$ itself is not OOI: indeed, multiplication by any integer n with $|n| \geq 2$ is injective.
(2) Every bounded group (in particular, $\mathbb{Z}_n$ for any integer $n \geq 2$) fails to be OI. Nevertheless, one checks easily that $\mathbb{Z}_3$, $\mathbb{Z}_4$, and $\mathbb{Z}_6$ are OOI groups. Observe that each of these groups has precisely two automorphisms.
(3) Every primary group is non-OI (see the characterization of torsion OI groups in [4]). Yet, for example, the 2-group $\mathbb{Z}_4$ is OOI.
(4) If G is an OI group and H is any Abelian group, then $G \oplus H$ is also OI (see [1]). In contrast, the OOI property is not preserved under such direct sums: for instance, $\mathbb{Z}_4 \oplus \mathbb{Z}$ is not OOI.

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