STRONGLY NOT DIVISIBLE ABELIAN GROUPS

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ABSTRACT. An Abelian group G is called strongly not divisible if $nG \neq G$ for every integer $n \neq \pm 1$. In [4], a characterization of the torsion strongly not divisible groups (referred to there as OI-groups) was obtained. In this note, we provide characterizations for broad classes of torsion-free and mixed strongly not divisible groups.

1. Introduction

Throughout, let G denote a nonzero Abelian group and let $\mathbb{Z}^*$ denote the set of all nonzero integers. As is well-known, G is *divisible* if $nG = G$ for any $n \in \mathbb{Z}^*$. Equivalently, G is not divisible if there is $n \in \mathbb{Z}^*$ such that $nG \neq G$. Since $nG = G$ clearly holds for $n \in \{\pm 1\}$, it is natural to ask: which Abelian groups satisfy $nG \neq G$ for every integer $n \notin \{-1, 0, 1\}$? In other words, which groups exhibit this stronger form of non-divisibility ?

A nonzero abelian group G is called an *OI-group* if $nG \neq G$ for every integer $n \notin \{-1, 0, 1\}$. Equivalently, if $\rho_n : G \to G$ denotes multiplication by n, then G is an OI-group ("onto-implies-invertible", see [3]) if and only if ρ_n is surjective only for $n = \pm 1$. In particular, G is an OI-group if and only if it is not p-divisible for any prime p. These are precisely the groups termed *strongly not divisible* in the title. However, for brevity, we continue to use the term OI-group throughout this paper.

In Section 2 we present some immediate yet useful properties related to the OI property.

The paper [3] gives the definition of OI modules and provides two (non)examples of Abelian OI-groups: $\mathbb{Q}$ is not OI and $\bigoplus_{p \in \mathbb{P}} \mathbb{Z}_p$ is OI.

In another work [4], a characterization was given for OI *torsion* Abelian groups: *A torsion Abelian group G is OI iff for every prime p, the group G has a cyclic summand of order p^s for some positive integer s.* However, there are some issues with the proof of this result, corrected in Section 3. Section 4 focuses on the well-studied classes of torsion-free OI groups, whereas Section 5 investigates how the OI property extends to mixed groups. In Section 6, we study how the OI property interacts with the classical functors in homological algebra, and, in the final section, two approaches are presented to describe the OI-groups using cones generated by certain natural choices of generators..

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2. Prerequisites

2.1. **OI properties.** We use without proof the well-known equality, $n(\bigoplus_{i\in I} G_i) = \bigoplus_{i\in I} nG_i$, for groups G_i and integers n.

In this subsection, we gather some simple yet useful results related to the OI property.

Proposition 2.1. *Divisible groups are not OI (and OI groups are not divisible).*

Proof. Follows from definitions. □

Proposition 2.2. *For any prime p, the p-groups are not OI.*

Proof. Indeed, for any prime $q \neq p$ and any p-group G, $qG = G$, so G is not OI. □

Recall that a subgroup H is called *pure* in a group G if, for every integer n, $nH = H \cap nG$. Next, we use the well-known fact that pure subgroups of divisible groups are divisible.

Theorem 2.3. *Let H be a pure subgroup of a group G. If H is OI then also G is OI.*

Proof. Assume G is not OI. There exists an integer $n \notin \{0, \pm 1\}$ such that $nG = G$. Hence, by purity, $nH = H \cap nG = H \cap G = H$, that is, the pure subgroup H is not OI. □

Corollary 2.4. *Let $G = H \oplus K$. If H (or K) is OI then G is OI.*

Proof. Follows from the previous theorem. □

Corollary 2.5. *A group is OI iff its reduced part is OI.*

Proof. If $D(G)$ is the maximal divisible subgroup of G, then $G = D(G) \oplus R$, where, up to isomorphism R is the reduced part of G. □

In particular, a direct sum is OI, if at least one summand is OI, that is, the *OI property is* (actually, more than) *preserved by direct sums*.

The converse (the OI property passes to direct summands) fails as witnessed by the following torsion-free

Example. For any prime p, denote $\mathbb{Z}^{(p)} := \{\frac{m}{p^k} : m \in \mathbb{Z}, k \geq 0\}$ the well-known (rank 1) rational group. Take $G = H \oplus K$ with $H = \mathbb{Z}^{(2)}$ and $K = \mathbb{Z}^{(3)}$. In Section 5, we show that H, K are not OI but G is OI (and a more general result).

Remarks. 1) Moreover, the OI property does **not** pass to *fully invariant* direct summands. As an **example**, let $G = \mathbb{Q} \oplus \mathbb{Z}$. Since $\mathbb{Z}$ is OI, it follows by the previous corollary that G is OI. As $Hom(\mathbb{Q}, \mathbb{Z}) = 0$, $\mathbb{Q}$ is a (divisible) fully invariant direct summand of G which is not OI.

2) *The OI and the reduced properties are independent.* Indeed, $\mathbb{Q} \oplus \mathbb{Z}$ is OI but not reduced and the cyclic group $\mathbb{Z}_p$ is reduced but not OI.

2.2. **OI and factor groups.** For a prime number p and a subgroup H of a group G, we have $p(G/H) = (pG + H)/H$.

Lemma 2.6. *(i) If G/H is OI, so is G;*
(ii) If G/H is not OI, then G may be OI.

Proof. (i) If $pG = G$ then $p(G/H) = (pG + H)/H = G/H$. Hence, if G is not OI then G/H is not OI.

(ii) According to the formula above, in general $pG + H = G$ does not imply $pG = G$. As an example, $G = H \oplus K$, with $H = \mathbb{Z}^{(2)}$ and $K = \mathbb{Z}^{(3)}$, is OI, but $G/K = (H + K))/K \cong H$ is not OI. □

2.3. **Hopfian and OI are independent.** It is easy to see that if x is invertible in R and M_R is an R-module, then $\rho_x : M \to M$ is an isomorphism. Thus, *one could consider* OI modules as a generalization of Hopfian modules, that is, the class of modules in which every surjective endomorphism is an automorphism.

In general the *Hopfian property does not imply the OI property.* As an example, consider the Abelian group $\mathbb{Z}_2$ and $3 \in \mathbb{Z}$. The multiplication with 3 is $1_{\mathbb{Z}_2}$, so an automorphism, but 3 is not invertible in $\mathbb{Z}$.

As a torsion-free example, consider $\mathbb{Z}^{(p)}$. The multiplication by p is an automorphism but p is not invertible in $\mathbb{Z}$.

The obstruction, when someone attempts to prove 'Hopfian ⇒ OI', is that

$$\rho_x \text{ automorphism} \Rightarrow x \text{ is a unit}$$

fails, in general.

Since the definition of OI refers only to multiplications (which are special endomorphisms), clearly *OI does not imply Hopfian.*

3. The torsion OI-groups

Recall from [4] the following characterization.

Theorem 3.1. *Let G be a torsion group. Then G is an OI-group iff, for each prime p, G has a cyclic direct summand $\mathbb{Z}_{p^s}$, for some positive integer s.*

Proof. Let G be an OI-group and let $G = \oplus_p G_p$ be its p-primary decomposition. Further decompose $G_p = R_p \oplus D_p$, where R_p is reduced and D_p is the maximal divisible subgroup of G_p.

By contradiction suppose $R_p = 0$, that is, G_p is p-divisible (i.e., $pG_p = G_p$). As for every prime $q \neq p$, we have $pG_q = G_q$, it follows that $pG = G$, a contradiction. Hence $R_p \neq 0$.

By Lemma 10.34 in [5], R_p contains a pure nonzero cyclic subgroup C. Since G is torsion, we can take $C = \mathbb{Z}_{p^s}$ for some s. By Kulikov (see **27.5** in [2]), a pure subgroup of bounded order is a direct summand. Consequently, C is a direct summand of R_p and so also of G_p and finally of G.

Conversely, suppose G has a direct summand $\mathbb{Z}_{p^s}$ for some s, for each prime p. If $p \mid G$, then p divides every direct summand of G. Since $p \nmid \mathbb{Z}_{p^s}$, it follows that $n \nmid G$ for every n divisible by p. So, if $n \mid G$, then $n \in \{-1, 1\}$ and G is an OI-group. □

Corollary 3.2. *Let $G = \bigoplus \mathbb{Z}_p$ where p runs over all primes. Then G is an OI-group.*

This is example 1.3 in [3].

Remarks. 1) The central argument (highlighted in the above proof) appeared in the proof given in [4], as follows: *if G is OI then G_p is OI.* This reasoning fails for two reasons already noted in the previous section: the OI property does not generally pass to direct summands, and, no p-group is OI. Fortunately, the proof can be corrected (as shown above) and the statement remains valid.

2) According to Corollary 2.4, if, for any torsion OI-group, we directly add any other torsion group (OI or not), it still has the characterization property (i.e., each p-primary component has a finite cyclic direct summand).

3) Lemma 10.34 from [5] follows also from **20** (C) and **27.2**, both in [2].

4) An alternative proof for the previous characterization theorem is given below.

Proposition 3.3. *A group $G = \bigoplus_{i\in I} G_i$ ($G = \prod_{i\in I} G_i$) is OI iff for each prime p there exists $i \in I$ with $pG_i \neq G_i$.*

Proof. It follows from the definition that the conditions are necessary. These are also sufficient, since if for each p there exists $i \in I$ with $pG_i \neq G_i$ then and $pG \neq G$, i.e., G is OI. □

Corollary 3.4. *A non-zero torsion group $T = \bigoplus_{p\in\Pi} T_p$, where T_p is a non-zero p-component of T and $\Pi \subseteq \mathbb{P}$, is OI iff $\Pi = \mathbb{P}$ and $pT_p \neq T_p$ for all $p \in \Pi$.*

Remark. Recall that a p-group G is divisible iff all its order p elements have infinite height ([2] §20), and each element of order p and finite height can be embedded in a finite cyclic direct summand. From this, the above characterization of the torsion OI groups (from [4]) follows.

In certain special cases, simpler proofs are available.

Proposition 3.5. *Bounded groups (in particular finite groups) are not OI.*

Proof. A group G is *bounded* if there exists $n \in \mathbb{Z}^*$ such that $nG = 0$. Then $(n+1)G = G$, so G is not OI. □

In particular, we get example 2.2 in [4]: for any integer $n \geq 2$, $\mathbb{Z}_n$ is not OI.

Note that $G = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2} \oplus ... \oplus \mathbb{Z}_{p^n} \oplus ...$ is an unbounded p-group which is not OI.

4. The torsion-free OI-groups

First, recall some previously given examples: the finite rank free groups are OI and, the divisible torsion-free groups are not OI.

4.1. **The completely decomposable torsion-free groups.** The following characterization follows directly from Proposition 3.3.

Corollary 4.1. *A completely decomposable torsion-free group G is OI iff for each prime p there exists a non p-divisible homogeneous component of G.*

Recall that the type $\mathbf{t}(\mathbb{Z}^{(p)}) = (0, ..., 0, \infty, 0...)$, whence $p\mathbb{Z}^{(p)} = \mathbb{Z}^{(p)}$. Thus, being p-divisible, $\mathbb{Z}^{(p)}$ is not OI, for any prime p.

More generally,

Corollary 4.2. *(i) If a rank one torsion-free group has an ∞ in its type, it is not OI.*

(ii) Let $p \neq q$ be primes. Both $\mathbb{Z}^{(p)}$ and $\mathbb{Z}^{(q)}$ are not OI but $G = \mathbb{Z}^{(p)} \oplus \mathbb{Z}^{(q)}$ is OI.

However, if $\mathbf{t}(H) \neq \mathbf{t}(K)$ then $G = H \oplus K$ fails to have the property OI.

Example. Take the group $H = \mathbb{Z}^{(2)}$ of rank 1 and type $\mathbf{t}_1 = (\infty, 0, 0, ...)$, and the group K of rank 1 and type $\mathbf{t}_2 = (\infty, \infty, 0, ...)$. Then $\mathbf{t}_1 < \mathbf{t}_2$, but H and K are p_1-divisible, so $G = H \oplus K$ is also p_1-divisible (here $p_1 = 2$).

4.2. **Separable groups.** Recall that a torsion-free group G is called *separable* if every finite subset of G is contained in a completely decomposable direct summand of G. According to Theorem 2.4, a separable group G is OI whenever a completely decomposable summand is OI. Building on Proposition 3.3 we get

Theorem 4.3. *Let G be a separable group. If there exist a finite subset of G, contained in an OI completely decomposable summand, then G is OI.*

More precisely

Corollary 4.4. *A separable torsion-free group G is OI iff for each prime p, there exists a non p-divisible rank 1 direct summand in G.*

Proof. The conditions are clearly sufficient. Conversely, assume that there exists a prime p such that $pA = A$ for each rank 1 direct summand of G. Since G is separable, each $0 \neq x \in G$ is contained in some finite rank completely decomposable direct summand B. From hypothesis it follows that $pB = B$. In particular, the p-height of x is infinite, and since x was arbitrary, it follows that $pG = G$, a contradiction. □

5. The mixed OI-groups

First, using Theorem 2.3 we obtain some surprising results.

Proposition 5.1. *Let G be a mixed group. Then:*
(i) G is OI whenever its torsion part $T(G)$ is OI.
(ii) If $G/T(G)$ is OI, so is G.

Proof. (i) Follows from Theorem 2.3.
(ii) Follows from Lemma 2.6. □

None of both converses hold. For (i) take $\mathbb{Z} \oplus \mathbb{Z}_2$, and for (ii) we need a non-splitting mixed example: $G = \prod_p \mathbb{Z}_p$. As direct product of reduced groups, G is reduced. Then $G/T(G)$ is divisible so not OI, but G is OI according to Proposition 3.3.

More precisely

Theorem 5.2. *A mixed group G is OI iff for each prime p, the condition $pT(G) = T(G)$ implies $p(G/T(G)) \neq G/T(G)$.*

Proof. One way, suppose G is OI and $pT(G) = T(G)$. Then $p(G/T(G)) \neq G/T(G)$ since $pG \neq G$.

Conversely, assume that for every p, if $pT(G) \neq T(G)$. Then $pG \neq G$ by the purity of $T(G)$. If $pT(G) = T(G)$ then $pG \neq G$ because $p(G/T(G)) \neq G/T(G)$, i.e., G is OI. □

6. Special constructions

In this section, we investigate how the OI property interacts with the classical functors in homological algebra.

Recall that if $pA = A$ or $pB = B$ then $p(A \otimes B) = A \otimes B$ (see [2] §59).

Proposition 6.1. *If A and B are OI then also $A \otimes B$ is OI.*

Proof. We can suppose that A and B are reduced. If A has a cyclic direct summand $\mathbb{Z}_{p^s}$ for some $s \geq 0$ then, as $pB \neq B$, by the properties of tensor product (see [2], §59) $A \otimes B$ has a direct summand $\mathbb{Z}_{p^s} \otimes B \cong B/p^sB$. So $A \otimes B$ is not p-divisible for each $p \in \Pi_1 \cup \Pi_2$, where $\Pi_1 = \{q \in \mathbb{P} \,|\, A_q \neq 0\}$, $\Pi_2 = \{q \in \mathbb{P} \,|\, B_q \neq 0\}$. Assume $p \in \mathbb{P} \setminus (\Pi_1 \cup \Pi_2)$. Then $p(A/T(A)) \neq A/T(A)$ and $p(B/T(B)) \neq B/T(B)$ since $T(A)$, $T(B)$ are p-divisible, but A, B is not p-divisible. So there exist $a \in (A/T(A)) \setminus p(A/T(A))$, $b \in (B/T(B)) \setminus p(B/T(B))$ and so by [2], Exercise 9, §60, $a \otimes b \in \Big((A/T) \otimes (B/T(B))\Big) \setminus p\Big(\big(A/T(A)\big) \otimes \big(B/T(B)\big)\Big)$. Together with the purely exact sequence

$$0 \to T(A) \to A \to A/T(A) \to 0,$$

we have the purely exact sequences

$$0 \to T(A) \otimes B \to A \otimes B \to (A/T(A)) \otimes B \to 0$$

and

$$0 \to T(A) \otimes (B/T(B)) \to A \otimes (B/T(B)) \to (A/T(A)) \otimes (B/T(B)) \to 0$$

(see [2], Theorem 60.4). According to the above, $(A/T(A)) \otimes (B/T(B))$ is not p-divisible, hence $A \otimes (B/T(B))$ and similarly $B \otimes (A/T(A))$ are not p-divisible. Hence $A \otimes B$ is not p-divisible. □

Proposition 6.2. *Let* $\mathrm{Tor}(A, B) \neq 0$. *Then* $\mathrm{Tor}(A, B)$ *is OI iff $A_p \neq 0$, $B_p \neq 0$ for each prime p and at least one of the groups A_p, B_p is not divisible.*

Proof. One way, since $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(T(A), T(B))$ then $T(A), T(B) \neq 0$. Moreover, if $\Pi_1 = \{p \in \mathbb{P} \,|\, A_p \neq 0\}$, $\Pi_2 = \{p \in \mathbb{P} \,|\, B_p \neq 0\}$ then $\Pi_1 \cap \Pi_2 \neq \emptyset$. Since $\mathrm{Tor}(A, B) \cong \bigoplus_{p \in \Pi_1 \cap \Pi_2} \mathrm{Tor}(A_p, B_p)$ and $\mathrm{Tor}(A_p, B_p)$ is p-group (see [2], §62) then $\Pi_1 \cap \Pi_2 = \mathbb{P}$, i.e. $\Pi_1 = \Pi_2 = \mathbb{P}$, so $A_p \neq 0$, $B_p \neq 0$ for each prime p. Since $\mathrm{Tor}(\mathbb{Z}(p^\infty), B) \cong B_p$, it follows that at least one of the groups A_p, B_p is not divisible.

Conversely, since $\mathrm{Tor}(\mathbb{Z}(p^\infty), B) \cong B_p$ and $\mathrm{Tor}(\mathbb{Z}_{p^m}, B) \cong B[p^m]$, by hypothesis it follows that $\mathrm{Tor}(A, B)_p \neq 0$, for every prime p. Moreover, since at least one of the groups A_p, B_p has a direct summand the type p^m, it follows that $\mathrm{Tor}(A, B)_p$ has a direct summand isomorphic to $A[p^m]$ or $B[p^m]$ and hence $\mathrm{Tor}(A, B)_p$ is not divisible. So $\mathrm{Tor}(A, B)$ is OI (see Corollary 3.4). □

Remarks. 1) The case of the functor *Hom* is more intricate. Depending on the structures of A and B, the group $Hom(A, B)$ may or may not be OI.

If A or B is divisible torsion-free group then $\mathrm{Hom}(A, B)$ also is divisible torsion-free group (see §43, [2]), so not OI. However, if $A = \bigoplus_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$ then $Hom(A, A) \cong \prod_{p \in \mathbb{P}} \widehat{\mathbb{Z}}_p$, where $\widehat{\mathbb{Z}}_p$ denotes the group of the p-adic integers (see Proposition 44.3 from [2]), i.e. $Hom(A, A)$ is OI. Also if $T(A)$ is OI and $B_p \neq 0$ for all p, since A has the direct summand of type $\mathbb{Z}_{p^s}$ with some integer $s \geq 0$ (depending on p) for each p, it follows that $Hom(A, B)$ has the direct summand isomorphic to $B[p^s]$ for each p (see §43, Example 2 from [2]). Hence $Hom(A, B)$ is OI.

2) Regarding the functor Ext, if B is a torsion-free group then $Ext(B, A)$ is divisible, so not OI for any A. But if $T(B)$ is OI and $A_p \neq 0$ for all p, then the group B has the direct summand of type $\mathbb{Z}_{p^s}$ for each p and so $Ext(B, A)$ has a direct summand isomorphic to $A[p^s]$ (see [2] §52). Hence, in this case, the group $Ext(B, A)$ is OI.

7. The description of OI-groups

Describing all OI-groups is not a straightforward task. Indeed, according to Corollary 2.4, if G is an OI-group and H is an arbitrary group, then the direct sum $G \oplus H$ is also an OI-group. Consequently, to achieve such a description, one must first identify a suitable class of "building blocks." A natural approach, therefore, is the following.

Definition. Let G, H be Abelian groups. Write $G \preceq H$ if G is isomorphic to a direct summand of H; i.e. there exists a group C such that $H \cong G \oplus C$. This relation is a *preorder* on isomorphism classes of Abelian groups. A class $\mathcal{P}$ of Abelian groups is said to be *upward closed* (for $\preceq$) if $G \in \mathcal{P}$ implies $(\forall H \ (G \preceq H \Rightarrow H \in \mathcal{P}))$. Equivalently, for all groups G, H, if $G \in \mathcal{P}$ then $G \oplus H \in \mathcal{P}$.

Every such property $\mathcal{P}$ can be expressed as a union of *cones*

$$\mathcal{P} = \bigcup_{G \in \mathcal{I}} \mathcal{C}(G),$$

where for a fixed group G,

$$\mathcal{C}(G) = \{\, H \mid G \preceq H \} = \{\, H \mid H = G \oplus C \text{ for some } C \,\}$$

and $\mathcal{I}$ is a class of $\mathcal{P}$-groups (the "building blocks") to be chosen.

If G decomposes as $G = G_1 \oplus G_2$, then $\mathcal{C}(G) \subseteq \mathcal{C}(G_1), \mathcal{C}(G_2)$.

In what follows, we present two attempts of describing the OI-groups in this manner.

7.1. The genuine OI-groups.

Due to Corollary 2.4, we can introduce the following

Definition. A group is called *genuine OI* if it indecomposable or has only OI (non-zero) direct summands.

Such OI-groups exist: $\mathbb{Z}$ and more generally finite rank free groups are genuine OI.

The genuine OI completely decomposable torsion-free groups and the genuine OI vector groups are describe at the end of this section.

However, this restriction does not align well with torsion OI-groups. In fact, in the class of torsion OI-groups, the groups are quite large, meaning, all their primary components must be nonzero.

Proposition 7.1. *Let G be a torsion group. If there exists a prime p with primary component $G_p = 0$, then G is not OI.*

Proof. Recall from [2], a property related to divisibility. For $a \in G$, we have $a \in nG$ whenever $\gcd(n, ord(a)) = 1$. Suppose $G_p = 0$. Then for every $a \in G$, $a \in pG$, that is, $pG = G$. Hence G is not OI. □

We are far for exhausting the (Abelian) OI-groups. However

Proposition 7.2. *(i) Any genuine OI-group is reduced and its summands are also OI.*

(ii) There are no torsion genuine OI-groups, and no mixed genuine OI-groups.

(iii) There are no algebraic compact genuine OI-groups.

Proof. (i) Obvious.

(ii) It suffices to recall Corollary 27.3 from [2]: if a group contains elements of finite order, then it has a cocyclic direct summand. Next, use the fact that any cocyclic group is not an OI-group.

(iii) Indeed, as reduced algebraic compact torsion-free groups contain a direct summand isomorphic to the group of p-adic integers, for some prime p, such groups are q-divisible for each prime $q \neq p$. □

Since there are no torsion nor mixed genuine OI-groups, next we focus on *torsion-free genuine OI-groups.*

The following simple result allows us to give numerous examples of completely decomposable and vector genuine OI-groups.

Proposition 7.3. *Let $G = \bigoplus_{\mathbf{t}\in\Omega} G_{\mathbf{t}}$ ($G = \prod_{\mathbf{t}\in\Omega} G_{\mathbf{t}}$), where Ω is a some set of types $\mathbf{t}$ and $G_{\mathbf{t}}$ is a direct sum of group of rank 1 and type $\mathbf{t}$ ($G_{\mathbf{t}}$ is a direct product of group of rank 1 and the type $\mathbf{t}$). The group G is genuine OI iff each $\mathbf{t}$ is not p-divisible for every prime p.*

Proof. The conditions are obviously necessary. Conversely, first for completely decomposable groups (say G), all direct decompositions are isomorphic [2] **86.1**, so every direct summand (say) A of G has a direct summand of rank 1 and type from Ω. Therefore, also A, as direct summand, is not p-divisible for every prime p.

Secondly, for vector groups, as any direct summand A of a vector group is also a vector group, it has some direct summand B of rank 1. By [2] **96.2**, the type of B is contained in Ω. As above, it follows that $pA \neq A$ for every prime p. □

Similarly, we can prove

Proposition 7.4. *A torsion-free separable group is genuine OI iff every rank 1 direct summand is not p-divisible for all primes p.*

7.2. **The minimal OI-groups.** We proceed with the following

Definition. Let $\mathcal{P}$ be a class of groups. A group G is called *minimal* $\mathcal{P}$-group if $G \in \mathcal{P}$ and G has no proper direct summand $H \in \mathcal{P}$.

Denote by $\mathcal{C}_1$ the *genuine* $\mathcal{P}$-groups, that is,

$\mathcal{C}_1 := \{G \in \mathcal{P}$: either G is indecomposable or every proper direct summand of G belongs to $\mathcal{P}\}$, and

by $\mathcal{C}_2$, the *minimal* $\mathcal{P}$-groups, that is,

$\mathcal{C}_2 := \{G \in \mathcal{P} : G$ has no proper direct summand $H \in \mathcal{P}\}$.

First notice that these two classes are incomparable and both contain the indecomposable OI-groups.

Some examples are: $\mathbb{Z}^{(2)} \oplus \mathbb{Z}^{(3)}$ is minimal OI but not genuine OI (clearly, not indecomposable), $\mathbb{Z} \oplus \mathbb{Z}$ is genuine OI but not minimal OI, $\mathbb{Z} \oplus \mathbb{Z}_{p^\infty}$ or $\mathbb{Z} \oplus \mathbb{Q}$ are in neither class.

Moreover

Proposition 7.5. $\mathcal{C}_1 \cap \mathcal{C}_2 = \{$ indecomposable $\mathcal{P}-$groups$\}$.

Proof. For any group G, every direct summand is OI or is not OI. So every decomposable group is not $\mathcal{P}$-minimal or not $\mathcal{P}$-genuine. □

Recall that the indecomposable groups are cocyclic or torsion-free. In our case, since cocyclic groups are not OI, we need to describe the *indecomposable torsion-free OI groups*.

The rank one indecomposable torsion-free groups (i.e., the rational (sub)groups), were classified in a previous section.

Note that there are uncountable many non-isomorphic indecomposable torsion-free groups (even of rank 2). Hence there are uncountable many corresponding cones.

Definition. A group G is called a *minimal* OI-group if G is OI and G has no proper OI-direct summand.

As already mentioned, the indecomposable OI-groups are minimal. It remains to describe *the decomposable OI-groups that are minimal.*

Remark. An OI-group G has the property that each proper (non-zero) subgroup is not OI iff G is torsion and all its p-components are simple p-groups, i.e., these are isomorphic to $\mathbb{Z}_p$.

Indeed, if G has of element a of infinite order then $\langle na\rangle$ for each integer n is OI and $\langle na\rangle \neq G$ for some n.

Torsion minimal OI

Each primary component must be minimal. According to Theorem 3.1, it must be cyclic. Hence

Proposition 7.6. *A torsion OI group is minimal iff for every prime p, the p-component is cyclic (i.e., reduced indecomposable).*

Mixed minimal OI

If G is *splitting* mixed, in order to be minimal OI, both $T(G)$ and $G/T(G)$ should **not** be OI.

Proposition 7.7. *A reduced mixed OI group G is minimal iff each p-component $T_p(G)$ is cyclic and for every relevant prime p (that is, $T_p(G) \neq 0$), $G/T_p(G)$ is p-divisible.*

Proof. The conditions are obviously necessary. Conversely, suppose $G = A \oplus B$ where $A, B \neq 0$. According to indecomposability, $T_p(G) \leq A$ or $T_p(G) \leq B$. If $0 \neq T_p(G) \leq A$ then by hypothesis $pB = B$, so B is not OI. □

Torsion-free minimal OI

Clearly, the indecomposable OI-groups are minimal OI. However, as already mentioned, $\mathbb{Z}^{(p)} \oplus \mathbb{Z}^{(q)}$ is decomposable minimal-OI.

For group G, let $\Pi(G) = \{p \in \mathbb{P} \,|\, pG \neq G\}$. Clearly, the minimal OI-groups are reduced.

Recall that torsion-free group G is called *quasi-homogeneous* if $\Pi(G) = \Pi(H)$ for each pure subgroup $0 \neq H \leq G$, i.e. the types of all non-zero elements of G have the symbol ∞ in the same components.

First note that if $G = \bigoplus_{i\in I} G_i$ and G is a minimal OI-group, then it has the following property
(*) : *for each $i \in I$ exist $p_i \in \mathbb{P}$ with $p_iG_i \neq G_i$ and $p_iG_j = G_j$ for all $j \in I \setminus \{i\}$.*

Indeed, if $\bigoplus_{j\in I\setminus\{i\}} G_j$ is not p-divisible for all $p \in \Pi(G_i)$, then $\bigoplus_{j\in I\setminus\{i\}} G_j$ also has the OI property. It follows that for a minimal OI-group $G = \bigoplus_{i\in I} G_i$, each G_i corresponds to only one prime p_i. In particular $|I| \leq \aleph_0$.

Proposition 7.8. *Let $G = \bigoplus_{i\in I} G_i$ be a reduced torsion-free group, where all G_i are quasi-homogeneous. Then G is minimal OI iff all G_i are indecomposable and satisfy the condition (*).*

Proof. One way, if $G_i = A_i \oplus B_i$ with $A_i, B_i \neq 0$ then $B_i \oplus \big(\bigoplus_{j\in I\setminus\{i\}} G_j\big)$ is OI by the quasi-homogeneous hypothesis.

Conversely, assume $Hom(G_i, G_j) \neq 0$ for some $i \neq j$. Then by (*), $qG_j \neq G_j$ and $qG_i = G_i$ for some $q \in \mathbb{P}$, so G_j has non-zero q-divisible subgroup, a contradiction. Hence each G_i is a fully invariant subgroup of G, so if $G = A \oplus B$, by the indecomposable hypothesis, each $G_i \leq A$ or $G_i \leq B$. Then it follows from (*) that $pB = B$ for some prime p with $pG_i \neq G_i$. Hence G is minimal OI. $\square$

As an example, $\bigoplus_{p\in\mathbb{P}} \mathbb{Z}_{(p)}$ is minimal OI-group, where $\mathbb{Z}_{(p)}$ denotes the group of all rational numbers with denominators coprime with p.

8. The pure version

By Theorem 2.3, in this section the direct summands are replaced by pure subgroups.

We only sketch the argument, since the corresponding 'building blocks' are more restrictive, and hence the procedure is less exhaustive.

8.1. **Pure-genuine.** An OI-group is called *pure-genuine* OI if all its nonzero pure subgroups are OI.

Clearly, every pure-genuine OI-group is genuine OI. As such, these groups are also torsion-free reduced.

The rôle of the indecomposables for genuine OI-groups is taken here by the *pure-simple* groups, that is, the groups G whose only pure subgroups are 0 and G.

As is well-known (e.g., see [1], **S 3.28**), these are the rank one groups.

More precisely

Proposition 8.1. *A torsion-free group G is pure-genuine OI iff G does not contain non-zero elements of p-divisible types for all prime p.*

Equivalently, $p^\omega G = \bigcap_{n\geq 1} p^n G = 0$ for all p, i.e. the type of non-zero elements of G does not contain the symbol ∞. Thus, each pure subgroup of rank 1 is OI.

8.2. **The pure-minimal OI-groups.** An OI-group G is called *pure-minimal* if it has no proper nonzero pure OI-subgroups.

As already mentioned,

$$\{\text{pure-minimal OI}\} \cap \{\text{pure-genuine OI}\} = \{\text{pure-simple OI}\} = \{\text{rank one OI}\}.$$

Since rank one torsion groups are not OI, this intersection is
$\{$torsion-free rank one OI$\}$.

Torsion pure-minimal

Since the cyclic p-groups are pure simple it follows that

Proposition 8.2. *A torsion OI-group is pure-minimal iff each of its p-components, for every prime p, is cyclic. In this case, the classes of pure-minimal and minimal OI-groups coincide.*

Torsion-free pure-minimal

Since the pure-simple groups are precisely the rank one groups, the *rank one* torsion-free OI-groups are pure-minimal.

Recall from Corollary 4.2, that if a rank one torsion-free group has an ∞ in its type, it is not OI.

Hence, only rank one torsion-free groups with no elements of infinite height are pure-minimal (these are also pure-simple).

Clearly, a quasi-homogeneous torsion-free OI-group is pure-minimal iff it has of rank 1.

More precisely

Lemma 8.3. *If $G = \bigoplus_{i \in I} G_i$ is a pure-minimal OI-group, where G_i are non-zero quasi-homogeneous torsion-free groups, then $|I| = 1$ and G has rank 1.*

Proof. If I is finite and $H = \langle \sum_{i \in I} x_i \rangle_*$ is the subgroup pure generated by some x_i, where $0 \neq x_i \in G_i$, then H is a OI-group since the groups G_i are quasi-homogeneous. If I is infinite, $i_1 \neq i_2 \in I$ and $H = \langle x_1 + x_2 \rangle_*$, where $0 \neq x_1 \in G_{i_1}$, $0 \neq x_2 \in G_{i_2}$, then $H \oplus \bigoplus_{i \in I \setminus \{i_1, i_2\}} G_i$ is a OI-group, a contradiction. □

Mixed pure-minimal

Proposition 8.4. *Let G be a mixed minimal OI-group, with $T = T(G)$. Then G is pure-minimal iff the following conditions hold:*

1) each T_p is cyclic, $T_p = 0$ at least for one prime p and if $T_p \neq 0$ then $p(G/T_p) = G/T_p$;

2) if H/T is a pure subgroup of G/T then $p(H/T) = H/T$, at least for one prime p with $T_p = 0$.

Proof. One way, if $p(H/T) \neq H/T$ for all p with $T_p = 0$, then $pH \neq H$ for such a p. Since T_q is non-zero cyclic for all the other primes q it follows that $qH \neq H$, as $T_q \leq H$. Hence H is proper pure OI-subgroup. That the other statements are necessary follows from Proposition 7.6.

Conversely, let H be a proper pure OI-subgroup of G. Then $H + T$ is also pure in G (see [2]; §26, Exercise 5). If $H + T = G$, then $T_p \not\leq H$ for some p and we show that $T_p \cap H = 0$. If $T_p \cap H \neq 0$ then $px = y \in H$ for some $T_p \ni x \notin H$. But $px = pz$ for some $z \in H$ whence $x - z \in T_p \cap H$ (because T_p is cyclic), so $x \in H$, a contradiction. Denote by $P_1 = \{p \in \mathbb{P} \,|\, T_p \cap H = 0\}$. If $T_1 = \bigoplus_{p \in P_1} T_p$ then $T_1 \cap H = 0$. Since $T_q \leq H$ for $q \in \mathbb{P} \setminus P_1$, it follows that $T + H = T_1 \oplus H$. Hence if $T + H = G$ then $T_1 \oplus H \cong G$ and $pH = H$ for all $p \in P_1$, which is impossible. Hence $T + H \neq G$. By hypothesis 2), $p((T + H)/T) = (T + H)/T$ for some p such that $T_p = 0$, since if $pT = T$ for such a p, then $p(T + H) = T + H$. Since $(T + H)/T \cong H/(T \cap H)$ and $T \cap H$ is p-divisible it follows that H is also p-divisible, which is impossible for an OI-group. □

In closing, we list three open problems, we were not able to solve.

1) Does there exist an indecomposable torsion-free pure-minimal OI-group that is not of rank one?

2) Do there exist groups $\{G_i : i \in I\}$ such that their direct sum $\bigoplus_{i \in I} G_i$ is a pure-minimal OI-group?

(Assume that the groups G_i are pairwise non-isomorphic.)

3) Does there exist a torsion-free group G such that $G \oplus H$ is a pure-minimal OI-group, for some rank-one torsion-free group H ?

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