A NILPOTENT 2×2 MATRIX THAT IS NOT SIMILAR TO ANY SCALAR MULTIPLE OF E_{12}.

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Recall the following result over Bézout domains.

Proposition 1. *Every nonzero nilpotent 2×2 matrix over a Bézout domain R is similar to rE_{12}, for some $r \in R$.*

Proof. Take $T = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ and $a^2 + bc = 0$ (with $a \neq 0$). We will construct an invertible matrix U such that $TU = U(rE_{12})$ with a suitable $r \in R$.

Let $x = \gcd(a, c)$ and denote $a = xy$, $c = xx'$ with $\gcd(y, x') = 1$. Then $x^2y^2 = -xx'b$ and since $\gcd(y, x') = 1$ implies $\gcd(y^2, x') = 1$, it follows x' divides x. Set $x = x'x''$ and so $T = \begin{bmatrix} x'x''y & x''y^2 \\ x'^2x'' & -x'x''y \end{bmatrix} = x'' \begin{bmatrix} x'y & -y^2 \\ x'^2 & -x'y \end{bmatrix} = x''T'$.

Since $\gcd(y, x') = 1$, there exist $s, t \in R$ such that $sy + tx' = 1$. Take $U = \begin{bmatrix} y & t \\ x' & -s \end{bmatrix}$ which is invertible (indeed, $\det(U) = -1$). One can check $T'U = \begin{bmatrix} 0 & y \\ 0 & x' \end{bmatrix} = UE_{12}$, so $r = x''$. □

Clearly, this does not rule out the possibility that the proposition could still hold—via a different proof—under the GCD hypothesis alone.

However, in the case of nilpotent 2×2 matrices, the statement cannot generally be proved over arbitrary GCD domains: as we demonstrate below, the Bézout hypothesis is, in fact, essential in fairly general settings.

Note that the unit U is constructed as $U = \begin{bmatrix} \frac{a}{\gcd(a,c)} & t \\ \frac{c}{\gcd(a,c)} & -s \end{bmatrix}$ with Bézout relation $s\frac{a}{\gcd(a,c)} + t\frac{c}{\gcd(a,c)} = 1$.

An attempt to construct a counterexample of 2×2 nilpotent matrix not similar to any multiple of E_{12}, would be to find two elements a, c with (coprime) $\frac{a}{\gcd(a,c)}$, $\frac{c}{\gcd(a,c)}$ and $c \mid a^2$.

This suggests the next result.

Proposition 2. *Let R be a GCD (commutative) domain that is not a Bézout domain. Suppose $a, b \in R$ are nonzero elements such that $\gcd(a, b) = 1$, but there do not exist $s, t \in R$ such that $sa + tb = 1$. Then the 2×2 nilpotent matrix $T = \begin{bmatrix} a^2b & -a^3 \\ ab^2 & -a^2b \end{bmatrix}$ is not similar to any scalar multiple of E_{12}.*

Proof. With $U = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbb{M}_2(R)$, from

$$TU = \begin{bmatrix} a^2b & -a^3 \\ ab^2 & -a^2b \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} a^2bx - a^3z & * \\ ab^2x - a^2bz & * \end{bmatrix} = \begin{bmatrix} 0 & rx \\ 0 & rz \end{bmatrix} = rUE_{12}$$

we get $ax = bz$. As $\gcd(a, b) = 1$ it follows that $a \mid z$, $b \mid x$. Write $x = bx'$, $z = az'$ and so $x' = z'$. Replacing in $xw-yz = 1$ gives $x'(bw+ay) = 1$, a contradiction. □

Examples. 1) $a = 2$, $b = X$ in $\mathbb{Z}[X]$. Here 2 and X in $\mathbb{Z}[X]$ have $\gcd(2, X) = 1$ but there are no $f, g \in \mathbb{Z}[X]$ such that $2f + Xg = 1$.

2) $a = X$, $b = Y$ in $k[X, Y]$ for any field k. Then $\gcd(X, Y) = 1$, and there do not exist s and t such that $sX + tY = 1$.