

Addendum to "3 × 3 idempotent matrices over some domains and a conjecture on nil-clean matrices"

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In several efforts to find a nilpotent 3 × 3 integral matrix that is not exchange, the study led to examples which all were exchange, with an exception which remained undecided.

Horia F. Pop constructed a small program which verifies any given 3 × 3 integral matrices for cleanness. Namely, for a given (nil-clean) 3 × 3 matrix A , the code browses incrementally the idempotent matrices E for which $\det(A - E) = \pm 1$, that is $A - E$ is invertible and so A is clean.

Revisiting the examples of the above mentioned paper, *all these nil-clean examples turned out to be clean !* Here is the list.

Page 100 $A = \begin{bmatrix} -13 & -25 & -39 \\ 1 & 2 & 4 \\ 4 & 8 & 12 \end{bmatrix}$. For $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\det(A - E) = -1$.

Page 102 $A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & -20 & -55 \\ 0 & 7 & 21 \end{bmatrix}$. For $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$ we have $\det(A - E) = 1$.

Page 103 $A = \begin{bmatrix} 0 & 4 & 6 \\ 0 & -20 & -50 \\ 0 & 7 & 21 \end{bmatrix}$. For $E = \begin{bmatrix} -5 & -5 & 3 \\ 6 & 6 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\det(A - E) = -1$.

Page 104 $A = \begin{bmatrix} -13 & -21 & -39 \\ 1 & 2 & 9 \\ 4 & 8 & 12 \end{bmatrix}$. This nil-clean matrix *was left undecided* whether exchange or not. Actually it is clean !

For $E = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\det(A - E) = -1$.

Page 105 $A = \begin{bmatrix} 53 & -5 & 7 \\ 241 & -25 & 29 \\ -212 & 20 & -28 \end{bmatrix}$ is even trivially clean, as $\det(A -$

$$I_3) = -1.$$

References

- [1] G. Călugăreanu 3×3 idempotent matrices over some domains and a conjecture on nil-clean matrices. Scientific Annals of "Al.I. Cuza" University, **68** (1) (2022), 91-106.