

Addendum to "Are nil-clean 3×3 integral matrices, exchange ?"

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In several efforts to find a nilpotent 3×3 integral matrix that is not exchange, the study led to examples which all were exchange, with an exception which remained undecided.

Horia F. Pop constructed a small program which verifies any given 3×3 integral matrices for cleanness. Namely, for a given (nil-clean) 3×3 matrix A , the code browses incrementally the idempotent matrices E for which $\det(A - E) = \pm 1$, that is $A - E$ is invertible and so A is clean.

Revisiting the examples of the above mentioned paper, *all these nil-clean examples turned out to be clean !* Here is the list.

Page 6 $A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -3 & 1 \\ 18 & -26 & 6 \end{bmatrix}$. For $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\det(A - E) = 1$.

Page 6 $A = \begin{bmatrix} -2 & 1 & 0 \\ -1 & -2 & 1 \\ 18 & -26 & 6 \end{bmatrix}$. For $E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we have (the same) $\det(A - E) = 1$.

Page 11 $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix}$. For $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ we have $\det(A - E) = 1$.

Page 11 Nilpotents $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & -4 & -1 \\ -15 & -23 & 3 \end{bmatrix}$. For $E = I_3$ we have $\det(A - E) = -1$.

and $A = \begin{bmatrix} 1 & 3 & -2 \\ -2 & -4 & 1 \\ 15 & 23 & 3 \end{bmatrix}$. For $E = I_3$ we have $\det(A - E) = -1$.

Actually, every nilpotent element, in any unital ring, is clean.

Indeed, if $t \in N(R)$ then $0 + t = 1 - (1 - t)$ are the nil-clean and clean decompositions.

References

- [1] G. Călugăreanu, H. F. Pop *Are nil-clean 3×3 integral matrices, exchange ?* Scientific Annals of "Al. I. Cuza" University, **67** (2) (2021), 171-186.