RINGS WHOSE IDEMPOTENTS FORM A SUBRING OR AN IDEAL

There are several papers describing rings for which the nilpotents form a subring (called NR rings; see [2]) or form an ideal (called NI rings; see [1]).

We could wonder what are the analogues for the set of all de idempotents of a ring.

Recently Y. Zhou ([3]) gave a simple characterization for the subring case.

Proposition 4.5. The following are equivalent for a ring R:

(1) R is an Abelian ring with $\mathrm{char}(R) = 2$.

(2) $idem(R)$ is a subring of R.

(3) $idem(R)$ is additively closed.

Proof. (1)⇒(2)⇒(3) The implications are clear.

(3)⇒(1) By (3), $2^2 = (1+1)^2 = 1+1 = 2$, showing that $2 = 0$. Let $e^2 = e \in R$. For $x \in R$, $ex(1-e) = e + (e + ex(1-e))$ is an idempotent by (3), so $ex(1-e) = 0$. Similarly, $(1-e)xe = 0$. It follows that $ex = xe$.

The ideal analogue is trivial.

Exercise. The set $idem(R)$ is a (left or right or) ideal of R iff R is Boolean.

Solution. 1) For every $r \in R$ and $1 \in idem(R)$, $r = r \cdot 1 = 1 \cdot r \in idem(R)$.

2) Alternatively, since $1 \in idem(R)$, $idem(R)$ contains a unit, so is the whole ring.

References

[1] Y. Chun, Y. C. Jeon, S. Kang, K. N. Lee, and Y. Lee. *A concept unifying the Armendariz and NI conditions*. Bull. Korean Math. Soc., **48** (1) (2011), 115-127.

[2] J. Šter *Rings in which nilpotents form a subring*. Carpathian J. Math. **32** (2) (2016), 251-258.

[3] Y. Zhou *Left uniquely generated elements in rings*. Comm. in Algebra **49** (9) (2021), 3825-3836.