IDEMPOTENT VON NEUMANN REGULAR RINGS

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Abstract. A unital ring R is called idempotent (von Neumann) regular if, for every element a in R, there exists an idempotent element e in R such that $a = aea$. An elementary proof demonstrates that these rings are precisely the Boolean rings, i.e., rings in which all elements are idempotent.

1. Introduction

Around 1935, John von Neumann introduced a class of rings, now widely known as von Neumann regular rings, during his work on continuous geometry and operator algebras. An element a of a ring R is called *(von Neumann) regular* if $a \in aRa$; that is, there exists $x \in R$ such that $a = axa$. The element x is sometimes referred to as an *inner inverse* for a. A ring R is called (von Neumann) regular if every element $a \in R$ is regular.

For simplicity, we will use the term regular exclusively to mean "von Neumann regular" throughout this discussion, whether referring to elements or rings.

One way to derive subclasses of regular rings is by imposing additional conditions on the inner inverse x. Since $0 = 0 \cdot x \cdot 0$ for any $x \in R$, the zero element poses no issues. However, the identity element $1 \in R$ warrants special attention. Regular rings are traditionally defined for unital rings, so we require an inner inverse for $a = 1$. In this case, $1 = 1 \cdot x \cdot 1 = x$. Thus, any additional property imposed on the inner inverses should naturally hold for the identity 1.

In 1968, Gertrude Ehrlich introduced the concept of unit-regular elements and rings, later expanding on it in 1976. She defined an element $a \in R$ as *unit-regular* if $a = aua$ for some unit $u \in U(R)$, where $U(R)$ denotes the set of all units in R. A ring R is called unit-regular if all its elements are unit-regular. Notably, this definition aligns with the requirement that 1, being a unit, satisfies the given property.

Beyond units, two other significant subsets of elements in ring theory are idempotents and nilpotents. This note explores a subclass of regular rings where the inner inverses are idempotents, motivated by the observation that 1 is not only a unit but also an idempotent. However, such a subclass cannot be defined using nilpotents as inner inverses, since 1 is not nilpotent in any nonzero ring.

We now define this subclass: An element a of a ring R is called *idempotent-regular* if there exists an idempotent $e \in R$ such that $a = aea$. A ring R is called idempotent-regular if all its elements are idempotent-regular. Clearly, all idempotents (including 0) are idempotent-regular. Furthermore, Boolean rings - rings in which every element is idempotent - are naturally idempotent-regular. The

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remainder of this note demonstrates that the idempotent-regular rings are precisely the Boolean rings.

2. The proof

It is well known (see [2] for proofs) that: Reduced rings (i.e., rings without nonzero nilpotents) are Abelian (i.e., all idempotents are central) and Abelian rings are Dedekind finite (i.e., one-sided inverses are two-sided). By char(R) we denote the characteristics of a ring R.

Step 1. *The only idempotent-regular unit is* 1.

Suppose $u = ueu$ with $u \in U(R)$ and $e^2 = e \in R$. Then $ue = 1 = eu$ so $e = u^{-1} = 1$ and consequently $u = 1$.

Step 2. *If R is idempotent-regular then $U(R) = \{1\}$ and char(R) = 2.*

From Step 1, it follows that $U(R) = \{1\}$. In any unital ring, -1 is also a unit, so $-1 = 1$, implying $2 = 0$. Thus char(R) = 2.

Step 3. *In a ring R with $U(R) = \{1\}$, every regular element is an idempotent.*

To prove this:

Notice that if r is a nilpotent element, then $1 + r$ being a unit implies $r = 0$. Hence, R is a reduced ring, and hence an Abelian ring.

For any regular element $a = axa$, the idempotents ax, xa are central.

This implies $a = a^2x = xa^2$. For the central idempotent $e = ax$, consider $u = ex + (1 - e)$, $v = ea + (1 - e)$. Since $ea = ae$, $vu = eaex + (1 - e)^2 = e^2ax + (1 - e) = e + (1 - e) = 1$. As reduced rings are Dedekind finite, $uv = 1$, so $u = v^{-1} \in U(R)$. From $a = a^2x = ae$ it follows that $a(1 - e) = 0$. Thus, $aua = a(ex + (1 - e))a = axa = a$. Finally, since u is a unit, it must be 1 and so $a = a^2$.

Step 4. *The idempotent-regular rings are precisely the Boolean rings.*

Since any idempotent-regular ring is regular, the statement follows from earlier steps.

In a footnote of P.M. Cohn's 1958 paper, "Rings of Zero-Divisors" (p. 913), it is observed: "M. P. Drazin has proved, somewhat more generally, that in any (not necessarily commutative) ring with 1 and no other invertible elements, every regular element is idempotent," i.e., Step 3 above. As we were unable to locate a direct reference, we have provided an elementary proof.

Remark. The above proof "hides" some additional facts: if ax and xa are central, then a is *strongly regular* (i.e., $a = a^2x = xa^2$) and therefore unit-regular. We avoided introducing strongly regular elements and their connection to unit-regularity to keep the proof elementary.

References

[1] P. M. Cohn *Rings of zero-divisors*. Proc. Amer. Math. Soc. **9** (1958), 909-914.
[2] K. R. Goodearl *Von Neumann regular rings*. Second edition, Krieger Publishing Co., 1991

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