

Regular elements lift modulo nil ideals

Grigore Călugăreanu

November 12, 2025

1 Introduction

It is well-known that idempotents (and units) can be lifted modulo every nil ideal of an arbitrary ring (see [4] **21.28**).

As idempotents and units are regular elements in any ring, a natural question (and possible generalization) is: can regular elements be lifted modulo every nil ideal of an arbitrary ring? The answer is affirmative, but I dare say this result is not widely known.

In what follows, we present two proofs selected from the existing literature. Thanks are due to T. Y. Lam and P. P. Nielsen for pointing out these references.

2 Via strongly lifting

From [3]

An one-sided ideal I of a ring R is said to be *strongly lifting* if whenever $x^2 - x \in I$ for some $x \in R$, there is an idempotent $e \in xR$ such that $e - x \in I$.

Theorem 4.9. *If a one-sided ideal I of a ring R is strongly lifting, then regular elements lift modulo I .*

From [6]

An one-sided ideal is called π -regular if some power of each element is regular (i.e., for each $a \in I$ there exist a positive integer n and an $x \in R$ such that $a^n = a^n x a^n$). The following result is due to Menal [[5], Lemma 3] in the case of two-sided ideals.

Proposition 7. *Every π -regular right ideal (and hence every nil right ideal) is strongly lifting.*

3 Via lifting ideal

From [2]

We define the notion of $x \in R$ being regular modulo a left ideal I (meaning $x - xyx \in I$ for some $y \in R$), and being *liftable to a regular element modulo I* (meaning $z - x \in I$ for some $z \in \text{reg}(R)$).

An ideal $J \subseteq R$ is a *lifting ideal* if idempotents lift modulo every left ideal I contained in J . For instance, *nil ideals* J are *always lifting*, as we can easily check by a slight modification of the standard argument given in [[4] (21.28)].

9.1 An ideal J is a lifting ideal in a ring R iff $x \in R$ is such that $x^2 - x \in J$, then there exists an idempotent $e \in Rx$ such that $1 - e \in R(1 - x)$.

(See Lemma 2.3 and Proposition 4.3 in [1]).

and using **9.1** the following result is proved.

Theorem 9.3. *Let J be any lifting ideal in a ring R . Then regular elements lift modulo every left ideal $I \subseteq J$.*

4 Remark

In [3], lifting ideals are called *fully lifting* and it is proved that

Proposition 5.6. *Any fully lifting left ideal is strongly lifting.*

References

- [1] D. Khurana, R.N. Gupta *Lifting idempotents and projective covers*. Kyungpook Math. J. **41** (2001) 217-227.
- [2] D. Khurana, T. Y. Lam *Rings with internal cancellation*. J. of Algebra **284** (2005) 203-235.
- [3] D. Khurana, T. Y. Lam, P. P. Nielsen *An ensemble of idempotent lifting hypotheses*. J. Pure Appl. Algebra **222** (6) (2018), 1489-1511.
- [4] T. Y. Lam *A First Course in Noncommutative Rings*. Graduate Texts in Mathematics, Vol. 131, 2nd edn. (Springer, 2001).
- [5] P. Menal *On π -regular rings whose primitive factor rings are artinian*. J. Pure Appl. Algebra **20** (1981), 71-78.
- [6] W. K. Nicholson, Y. Zhou *Strong lifting*. J. Algebra **285** (2) (2005), 795-818.