

# An exchange useful trick

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An elementary trick, more or less always used in the context of exchange rings, is the following:

**Exercise 1** For a ring  $R$  and elements  $a, e \in R$ , if  $e \in Ra$  is an idempotent, an element  $b \in R$  can be chosen such that  $e = ba$  and  $bab = b$ .

Note that such an element  $b$  is (von Neumann) regular.

**Solution 2** If  $e \in Ra$  there exists  $r \in R$  such that  $e = ra$ . If  $e$  is idempotent then  $rara = ra$  which does not always imply  $rar = r$ .

If  $rar \neq r$  we replace  $r$  by  $b = rar$ . Then  $e = rara = ba$  and  $bab = rararar = rar = b$ , as desired.