Exercise 37, p. 427 in EXCURSIONS: $\begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}^4$ is not clean

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For $A = \begin{bmatrix} 3 & 9 \\ -7 & -2 \end{bmatrix}$, we know now that A and A^3 are not clean but A^2 is clean. In the exercise mentioned it is asked: "this leaves one to wonder whether it is true that A^n is clean iff n is even".

We show that A^4 *is not clean* using the characterization theorem for (non-trivial) clean 2×2 integral matrices.

We have $A^4 = \begin{bmatrix} 2853 & -1017 \\ 791 & 3418 \end{bmatrix}$ and $\det(A^4) = 10556001 = 57^4$.

The quadratic Diophantine equations are

$$-1017x^2 + 565xy - 791y^2 + 1071x + (-10553148 \pm 1)y = 0 \quad (\pm 3).$$

Both equations have only two solutions, namely $(0, 0)$ and $(1, 0)$ to be replaced in the linear Diophantine equations

$$-565x + 791y - 1017z + 10553148 = \pm 1 \quad (\pm 2).$$

Replacing $(0, 0)$ gives no integer z for

$$-1017z + 10553148 = \pm 1$$

and similarly for $(1, 0)$

$$-565 - 1017z + 10553148 = \pm 1.$$

Hence A^4 is not clean.