Elementary divisor ring: the definition

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Abstract

Kaplansky required in his definition of diagonal reduction (used for defining the elementary divisor rings), each diagonal entry to divide the next entry below. For commutative rings, this requirement is superfluous.

As early as 1949, Kaplansky (see [3]) gave the following

Definition 1. We say that the matrix A admits *diagonal reduction* if there exist unimodular matrices P, Q such that $PAQ = diag(d_1, d_2, ...)$ where d_i is a divisor of d_{i+1}. If every matrix over R admits diagonal reduction, we call R an *elementary divisor ring*.

If every 1×2 and 2×1 matrix over R admits diagonal reduction then R is an *Hermite* ring.

Also recall that a ring is a *Bezout* ring if every finitely generated ideal is principal.

Obviously *an elementary divisor ring is Hermite* and it is easy to see that *an Hermite ring is Bezout.* Examples that neither implication is reversible are provided in [1].

T. Y. Lam, private communication (20.07.2020)**:** "As for "elementary divisor rings", it is perhaps not 100% correct to say that we're using Kaplansky's definition. The fact is, that Kaplansky considered the general case of rectangular matrices, and his definition involved some rather technical requirements on the "normal form" of a matrix. Instead of following Kaplansky 100%, modern workers usually prefer to use Henriksen's definition, which required the existence of a diagonal normal form *for any square matrix* without any requirements on the diagonal entries".

Indeed, the following definition is more general.

Definition 2. A ring R is an elementary divisor ring, if every matrix over R is equivalent to a diagonal matrix.

However, it is not more general if the ring is supposed to be commutative.

This follows from [4] (where throughout *rings are assumed commutative*).

(3.1) Theorem. *All diagonal matrices over a ring R admit diagonal reduction if and only if R is a Bezout ring.*

(3.7) Corollary. *The ring R is an elementary divisor ring if (and only if) every 2×2 matrix over R is equivalent to a diagonal matrix.*

The rings considered in [2] are also supposed to be commutative.

References

[1] L. Gillman, M. Henriksen *Some remarks about elementary divisor rings*. Trans. Amer. Math. Soc. **82** (1956), 362-365.

[2] M. Henriksen *Some remarks on elementary divisor rings*. II, Michigan Math. J. **3** (1955/56), 159-163.

[3] I. Kaplansky *Elementary divisors and modules*. Trans. Amer. Math. Soc. **66** (1949), 464-491.

[4] M. D. Larsen, W. J. Lewis, T. S. Shores *Elementary divisor rings and finitely presented modules*. Trans. Amer. Math. Soc. **187** (1) (1974), 231-248.