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CORRIGENDUM AND ADDENDUM TO THE PAPER "SIMILARITY FOR 2×2 MATRICES OBTAINED BY CLOCKWISE ROTATION"

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Corrigendum and Addendum

ABSTRACT

A computational error was found in the above-mentioned paper. Although the error seemed minor at first and did not appear to alter the main conclusions, a more careful re-evaluation has uncovered additional insights into the structure and properties of rotatable 2×2 matrices.

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1. INTRODUCTION

One year after the publication of this paper in Mathematica Pannonica, it was brought to my attention that the computation of $\det(M)$, provided in the Appendix of the paper, contained an error. The correct expression is $2c(a-b)(b-d)(c-d)$ not $2(c-b)(b-d)(c-d)$ as originally stated. Furthermore, when $ad = bc$ and $a + d = b + c$ we actually have $\det(M) = 0$.

The impact of this correction (from $2(c-b)(b-d)(c-d)$ to $2c(a-b)(b-d)(c-d)$) is minor. In Theorem 2.4, it merely requires adjusting the list of cases leading to $\det(M) = 0$, from $b = c$ or $b = d$ or $c = d$ or $\mathrm{char}(R) = 2$, to $b = c$ or $b = d$ or $a = b$ or $c = 0$ or $\mathrm{char}(R) = 2$. The additional case $c = 0$ (which implies $a = 0$ or $d = 0$) is straightforward, while the new case $a = b$ is analogous to the former case $b = c$.

These cases were considered to ensure $\det(M) = 0$ and were analyzed in detail in the original paper (see pages 195–198 and 199–203) , which therefore require no modification.

However, the additional observation that $\det(M) = 0$ always holds – not only in the specific cases listed above – has a significant impact. It changes the interpretation of the results presented in the paper: instead of providing a complete description of all rotatable matrices over integral domains, the original results now correctly characterize only certain specific examples.

Building on this new insight, we present here a genuine continuation of the previous work, which leads toward a possible full description of all rotatable matrices over integral domains. All the results obtained in Section 2 are new.

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2. THE ROTABLE MATRICES

For reader's convenience we start with a useful calculation.

LEMMA. Suppose $ad = bc$ and $a + d = b + c$. Then $(b - d)(c - d) = (a - b)(b - d) = (a - c)(c - d) = (a - b)(a - c) = 0$.

Proof. Simple computations using the given hypotheses. Here are two samples:
$(b - d)(c - d) = bc - cd - bd + d^2 = ad - cd - bd + d^2 = (a - c - b + d)d = 0$, or
$(a - b)(b - d) = ab - b^2 - ad + bd = ab - b^2 - bc + bd = (a - b - c + d)b = 0$. □

THEOREM. Let R be a commutative ring and let $A \in \mathbb{M}_2(R)$. There exists a nonzero matrix $B \in \mathbb{M}_2(R)$ such that $AB = Brot(A)$.

Proof. Denoting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & u \end{bmatrix}$, the matrix equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & u \end{bmatrix} = \begin{bmatrix} x & y \\ z & u \end{bmatrix}\begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

is equivalent to the homogeneous linear system

$$\begin{aligned} (a - c)x - dy + bz &= 0 \\ ax + (b - a)y - bu &= 0 \\ cx + (d - c)z - du &= 0 \\ cy - az + (d - b)u &= 0 \end{aligned} \qquad \text{(S)}$$

whose system matrix is

$$M = \begin{bmatrix} a - c & -d & b & 0 \\ a & b - a & 0 & -b \\ c & 0 & d - c & -d \\ 0 & c & -a & d - b \end{bmatrix}.$$

Using the necessary conditions $ad = bc$ and $a + d = b + c$, an elementary computation (see previous lemma) shows that $\det(M) = 2c(a - b)(b - d)(c - d) = 0$.

Recall, from Brown's book (see [1] 4.11 (e)), that the rank of an $n \times n$ matrix over a commutative ring R, defined using the annihilators of the determinantal ideals, is $< n$ iff its determinant is a zero divisor of R.

Also recall the famous N. McCoy's theorem (see [1] 5.3): Let $A \in \mathbb{M}_{m\times n}(R)$. The homogeneous (linear) system of equations $AX = 0$ has a nontrivial solution iff $rk(A) < n$.

As the determinant of the above matrix M turns out to be zero, it follows that the corresponding (linear) system (always) has a nontrivial solution. So, for any 2×2 matrix A, a nonzero matrix B with $AB = Brot(A)$, (always) exists. □

COROLLARY. Over any commutative ring R, the 2×2 matrix A is rotable iff the system (S) in the previous proof has a solution satisfying $xu - yz \in U(R)$. In particular, over fields, this holds iff $xu - yz \neq 0$.

Proof. Indeed, for A to be rotable, it is necessary and sufficient that the nonzero matrix B is invertible. This holds iff $\det(B)$ is a unit of R. □

Therefore, to determine all 2×2 rotatable matrices over a commutative ring, the following approach should be undertaken:

1) compute the rank r of M, which we already know it is ≤ 3,

2) choose r equations from the system (S) and form a nonhomogeneous linear system with r equations and r unknowns (among x, y, z, u),

3) explicitly solve this system,

4) find the conditions on a, b, c, d which assure that among the (existing, by the previous theorem) nonzero solutions x, y, z, u, there exists one with $xu - yz$ is a unit.

For instance, 1) in the above plan involves the computation of the 16, 3×3 minors of M, in order to give conditions when the rank is 3 or 2.

Surprisingly:

PROPOSITION. All the 16, 3×3 minors of the matrix M are zero. Hence $rk(M) \leq 2$.

Proof. Again simple computations. The easiest computations involve the 4 minors which include 3 of zeros on the secondary diagonal.

We give here another 3 samples. The parentheses before indicates, which rows respectively which columns were chosen.

$$(123,123)\begin{vmatrix} a-c & -d & b \\ a & b-a & 0 \\ c & 0 & d-c \end{vmatrix} = (a-c)(b-a)(d-c) - bc(b-a) + ad(d-c) = 0+0 = 0,$$

$$(134,123)\begin{vmatrix} a-c & b & 0 \\ a & 0 & -b \\ c & d-c & -d \end{vmatrix} = b[-bc + ad + (a-c)(d-c)] = b[0+0] = 0,$$

$$(234,234)\begin{vmatrix} b-a & 0 & -b \\ 0 & d-c & -d \\ c & -a & d-b \end{vmatrix} = (b-a)(d-c)(d-b) + bc(d-ca) - ad(b-a) = 0+0 = 0.$$

Alternatively, in a more ring theoretic way, consider the ideal $\langle ad-bc, a+d-b-c\rangle$. It can be shown that for each of the 16, 3×3 minors m, there exists polynomials $q_1, q_2 \in \mathbb{Z}[a,b,c,d]$ such that $m = q_1(ad-bc) + q_2(a+d-b-c)$. Hence all order 3 minors vanish and $rk(M) \leq 2$. □

REMARK. Over an arbitrary commutative ring it is not easy to characterize the cases in which $rk(M) = 1$. By contraposition this would also describe when $rk(M) = 2$.

A simple example of a matrix with $rk(M) = 1$ occurs when three of the entries a, b, c, d are zero and the remaining one is nonzero but has zero square (if zero divisors are allowed).

This difficulty motivated the assumption of the integral domain hypothesis in the original paper.

To continue with the general plan mentioned above, we now assume $rk(M) = 2$. We select two principal equations and two principal unknowns, solve the resulting system and determine the sufficient conditions (recall that the necessary ones are $ad = bc$, $a+d = b+c$) for the existence of a nonzero solution, in which $xu - yz$ is a unit. Even in the case of an integral domain, this involves not trivial work. In what follows, several simple sufficient conditions for a matrix A to be rotable are established.

The simplest choices (out of the 36 possibilities !) of two principal equations and two principal unknowns, are $(34,12)$ and $(12,34)$. The left two figures indicate the choice of the rows and the right figures, the choice of the columns.

In the $(34,12)$ choice, if c is a unit we get the solutions $x = [(c-d)z + du]c^{-1}$ and $y = [az + (b-d)u]c^{-1}$, and we have to find values for z and u in order to have $xu - yz \in U(R)$. As c^{-1} is a unit, we can ignore the c^{-1} factor and look for values of z and u, such that $[(c-d)z + du]u - [az + (b-d)u]z$ is a unit.

An easy to handle choice is $z = u = 1$ for which $x = 1$, $y = (a+b-d)c^{-1}$. If $c-a-b+d$ is a unit, the matrix A is rotable.

In the $(12,34)$ choice, if b is a unit we get the solutions $z = [(c-a)x + dy]b^{-1}$ and $u = [ax + (b-a)y]b^{-1}$, and we have to find values for x and y in order to have $xu - yz \in U(R)$. As b^{-1} is a unit, we can ignore the b^{-1} factor and look for values of x and y, such that $x[(c-a)x + dy] - y[az + (b-d)u]z$ is a unit.

Again, an easy to handle choice is $z = u = 1$ for which $x = 1$, $y = (a+b-d)c^{-1}$. If $c-a-b+d$ is a unit, the matrix A is rotable.

This way, we have obtained the following result

PROPOSITION. Let R be a commutative ring and let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(R)$ such that $ad = bc$ and $a+d = b+c$. The matrix A is rotable if any of the following conditions are fulfilled.

(i) c and $c - a - b + d$ are units in R;
(ii) b and $c - a - b + d$ are units in R.

To be more explicit here are some details of the proof of (i).

As found in the paragraph before the proposition, the matrix U for $AU = Urot(A)$ is $U = \begin{bmatrix} 1 & y \\ 1 & 1 \end{bmatrix}$ where $y = (a + b - d)c^{-1}$.

(a) U is invertible.

Indeed, $\det(U) = 1 - y = 1 - (a + b - d)c^{-1} = (c - a - b + d)c^{-1} \in U(R)$ by hypotheses.

(b) We have to check four equalities.

(i) $a + b = c + dy$,
(ii) $ay + b = a + by$,
(iii) $c + d = c + d$, clear
(iv) $cy + d = a + b$, easy: $cy = a + b - d$.

For (i) and (ii) first observe that $a(b + c) = a^2 + bc$. Indeed, we just use $ad = bc$ and then $a(b + c) = a(a + d)$ follows from $a + d = b + c$.

For both equalities, we multiply (equivalently) by c, we eliminate $d = b + c - a$ and replace $b - d$ by $a - c$.

(i) $(a + b - c)c = (2a - c)(b + c - a)$ reduces to $2a(b + c) = 2(a^2 + bc)$.

(ii) $a(a + b - d)bc = ac + b(a + b - d)$ also reduces to $a(2a - c) + bc = ac + b(2a - c)$ and finally to $2(a^2 + bc) = 2a(b + c)$.

In summary, aside from the necessary adjustment to Proposition 2.4, all the results presented in the former paper remain correct. However, they no longer provide a complete description of all rotable matrices over integral domains; rather, they describe specific classes of rotable matrices of prescribed forms (including the distinction between cases with $\mathrm{char}(R) = 2$ and $\mathrm{char}(R) \neq 2$). To achieve a full characterization of rotable matrices over commutative rings, one would, in principle, need to examine all the remaining 34 possible configurations and determine the corresponding sufficient conditions – an exhaustive task that lies well beyond the scope of this note.

REFERENCES

[1] W. C. Brown *Matrices over commutative rings.* Marcel Dekker Inc., New York, Basel, Hong Kong 1993.