Journal of Pure and Applied Algebra 229 (2025) 107852

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Matrix invertible extensions over commutative rings. Part I: General theory

Grigore Călugăreanu [a], Horia F. Pop [b], Adrian Vasiu [c,*]

[a] Department of Mathematics, Babeş-Bolyai University, 1 Mihail Kogălniceanu Street, Cluj-Napoca 400084, Romania
[b] Department of Computer Science, Babeş-Bolyai University, 1 Mihail Kogălniceanu Street, Cluj-Napoca 400084, Romania
[c] Department of Mathematics and Statistics, Binghamton University, P. O. Box 6000, Binghamton, NY 13902-6000, USA

ARTICLE INFO

Article history:
Received 7 July 2024
Received in revised form 16 November 2024
Available online 4 December 2024
Communicated by A. Conca

MSC:
Primary: 15A83; 13G05; 19B10;
secondary: 13A05; 13F05; 15B33

Keywords:
Ring
Matrix
Projective module
Stable range
Unimodular

ABSTRACT

A unimodular 2×2 matrix with entries in a commutative R is called extendable (resp. simply extendable) if it extends to an invertible 3×3 matrix (resp. invertible 3×3 matrix whose $(3,3)$ entry is 0). We obtain necessary and sufficient conditions for a unimodular 2×2 matrix to be extendable (resp. simply extendable) and use them to study the class E_2 (resp. SE_2) of rings R with the property that all unimodular 2×2 matrices with entries in R are extendable (resp. simply extendable). We also study the larger class Π_2 of rings R with the property that all unimodular 2×2 matrices of determinant 0 and with entries in R are (simply) extendable (e.g., rings with trivial Picard groups or pre-Schreier domains). Among Dedekind domains, polynomial rings over $\mathbb{Z}$ and Hermite rings, only the EDRs belong to the class E_2 or SE_2. If R has stable range at most 2 (e.g., R is a Hermite ring or $\dim(R) \leq 1$), then R is an E_2 ring iff it is an SE_2 ring.

1. Introduction

Let R be a commutative ring with identity; we denote by $U(R)$ its group of units, by $J(R)$ its Jacobson radical, and by $Pic(R)$ its Picard group. For $m, n \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathbb{M}_{m\times n}(R)$ be the R-module of $m \times n$ matrices with entries in R; we view $\mathbb{M}_n(R) := \mathbb{M}_{n\times n}(R)$ as an R-algebra with identity I_n. Let $GL_n(R)$ be the general linear group of units of $\mathbb{M}_n(R)$, and let $SL_n(R) := \{M \in GL_n(R) | \det(M) = 1\}$ be the special linear subgroup of $GL_n(R)$.

* Corresponding author.
E-mail addresses: calu@math.ubbcluj.ro (G. Călugăreanu), horia.pop@ubbcluj.ro (H.F. Pop), avasiu@binghamton.edu (A. Vasiu).

https://doi.org/10.1016/j.jpaa.2024.107852
0022-4049/

For a free R-module F, let $Um(F)$ be the set of *unimodular* elements of F, i.e., of elements $v \in F$ for which there exists an R-linear map $L : F \to R$ such that $L(v) = 1$; we have an identity $U(R) = Um(R)$ of sets.

For $A \in \mathbb{M}_n(R)$, let A^T be its transpose, let $\mathrm{Tr}(A)$ be its trace, let $\chi_A(\lambda) \in R[\lambda]$ be its (monic) characteristic polynomial, and let $\nu_A := \chi'_A(0) \in R$. E.g., if $n = 3$, then $\chi_A(\lambda) = \lambda^3 - \mathrm{Tr}(A)\lambda^2 + \nu_A\lambda - \det(A)$.

As we will be using pairs, triples and quadruples extensively, the elements of R^n will be denoted as n-tuples except for the case of the R-linear map $L_A : R^n \to R^n$ defined by A, in which case they will be viewed as $n \times 1$ columns. Let Ker_A, Coker_A and Im_A be the kernel, cokernel and image (respectively) of L_A.

Recall that $B, C \in \mathbb{M}_{m\times n}(R)$ are said to be equivalent if there exist matrices $M \in GL_m(R)$ and $N \in GL_n(R)$ such that $C = MBN$; if $m = n$ and we can choose $N = M^{-1}$, then B and C are said to be similar. If all entries of $B - C$ belong to an ideal I of R, then we say that B and C are congruent modulo I. By the reduction of $A \in \mathbb{M}_n(R)$ modulo I we mean the image of A in $\mathbb{M}_n(R/I) \cong \mathbb{M}_n(R)/\mathbb{M}_n(I)$.

We say that $B \in \mathbb{M}_{m\times n}(R)$ *admits diagonal reduction* if it is equivalent to a matrix whose off diagonal entries are 0 and whose diagonal entries $b_{1,1}, \ldots, b_{s,s}$, with $s := \min\{m, n\}$, are such that $b_{i,i}$ divides $b_{i+1,i+1}$ for all $i \in \{1, \ldots, s-1\}$.

We will use the shorthand ‘iff’ for ‘if and only if’ in all that follows.

For $Q \in \mathbb{M}_3(R)$, let $\Theta(Q) \in \mathbb{M}_2(R)$ be obtained from Q by removing the third row and the third column. If $Q \in GL_3(R)$, then $\Theta(Q)$ modulo each maximal ideal of R is nonzero, hence $\Theta(Q) \in Um\big(\mathbb{M}_2(R)\big)$. The rule $Q \to \Theta(Q)$ defines a map

$$\Theta := \Theta_R : SL_3(R) \to Um\big(\mathbb{M}_2(R)\big)$$

and in this paper we study the image of Θ and in particular the class of rings R for which Θ is surjective (i.e., it has a right inverse).

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL_2(R)$, let $\sigma(M) := \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \det(M)^{-1} \end{bmatrix} \in SL_3(R)$. Clearly $\Theta\big(\sigma(M)\big) = M$, hence $GL_2(R) \subseteq Im(\Theta)$.

Definition 1.1. We say that a matrix $A \in \mathbb{M}_2(R)$ is SL_3-extendable if there exists $A^+ \in SL_3(R)$ such that $A = \Theta(A^+)$, and we call A^+ an SL_3-extension of A. If we can choose A^+ such that its $(3, 3)$ is 0, then we say that A is simply SL_3-extendable and that A^+ is a simple SL_3-extension of A.

As in this paper we do not consider SL_n-extensions with $n \geq 4$, in all that follows it will be understood that “extendable” means SL_3-extendable. Each extendable matrix is unimodular. Theorem 4.3 contains three other equivalent characterizations of simply extendable matrices; e.g., a unimodular 2×2 matrix admits diagonal reduction iff it is simply extendable.

The problem of deciding if $A \in \mathbb{M}_2(R)$ is simply extendable relates to classical studies of finitely generated stable free modules that aim to complete matrices in $\mathbb{M}_{n\times(n+m)}(R)$ whose $n \times n$ minors generate R (e.g., see [11]), but it fits within the general problem of finding square matrices with prescribed entries and coefficients of characteristic polynomials. Concretely, if $A \in Um\big(\mathbb{M}_2(R)\big)$ is simply extendable, its simple extensions A^+ have 5 prescribed entries (out of 9) and 2 prescribed coefficients (out of 3) of $\chi_{A^+}(\lambda) = \lambda^3 - \mathrm{Tr}(A)\lambda^2 + \nu_{A^+}\lambda - 1$; the nonempty subsets

$$\nu(A) := \{\nu_{A^+} | A^+ \text{ is a simple extension of } A\} \subseteq R$$

are sampled in Examples 4.5 and 7.3(1). Such general problems over fields have a long history and often complete results (cf. [4] and [7]).

In general Θ is not surjective (see Theorem 1.7(4) and Example 5.3).

If $\det(A) = 0$, then A is extendable iff it is simply extendable. More generally, a matrix $A \in \mathbb{M}_2(R)$ is extendable iff its reduction modulo $R\det(A)$ is simply extendable (see Lemma 4.1(1)). Based on this, Definition 1.1 leads to the study of 3 classes of rings to be named, in the spirit of [5] and [12], using indexed letters.

Definition 1.2. We say that R is:

(1) a Π_2 ring, if each $A \in Um(\mathbb{M}_2(R))$ with $\det(A) = 0$ is extendable;

(2) an E_2 ring, if each matrix in $Um(\mathbb{M}_2(R))$ is extendable (i.e., Θ is surjective);

(3) an SE_2 ring, if each matrix in $Um(\mathbb{M}_2(R))$ is simply extendable.

If moreover R is an integral domain, we replace ring by a domain (so we speak about Π_2 domains, E_2 domains, etc.)

We recall that R is called an *elementary divisor ring*, abbreviated as *EDR*, if all matrices with entries in R admit diagonal reduction. Equivalently, R is an *EDR* iff each matrix in $\mathbb{M}_2(R)$ is equivalent to a diagonal matrix and iff each finitely presented R-module is a direct sum of cyclic R-modules (see [10], Cor. (3.7) and Thm. (3.8); see also [20], Thm. 2.1).

Let $A \in Um(\mathbb{M}_2(R))$. If A admits diagonal reduction, let $M, N \in GL_2(R)$ be such that $MAN = \begin{bmatrix} 1 & 0 \\ 0 & \det(A) \end{bmatrix}$; for $A^+ := \sigma(M^{-1}) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \det(A) & 1 \\ 0 & -1 & 0 \end{bmatrix} \sigma(N^{-1})$ one easily computes $\Theta(A^+) = M^{-1}MANN^{-1} = A$ (this is a particular case of Lemma 4.1(2)), hence A is simply extendable. We conclude that:

Proposition 1.3. *Each EDR is an SE_2 ring.*

The following characterizations of Π_2 rings are presented in Section 5.

Theorem 1.4. *For a ring R the following statements are equivalent:*

(1) *The ring R is a Π_2 ring.*

(2) *Each matrix in $Um(\mathbb{M}_2(R))$ of zero determinant is non-full, i.e., the product of two matrices of sizes 2×1 and 1×2 (equivalently, the R-linear map L_A factors as a composite R-linear map $R^2 \to R \to R^2$).*

(3) *For each matrix in $Um(\mathbb{M}_2(R))$ of zero determinant, one (hence both) of the R-modules Ker_A and Im_A is isomorphic to R.*

(4) *Each projective R-module of rank 1 generated by two elements is isomorphic to R.*

For pre-Schreier domains see Section 2. Recall that an integral domain R is a pre-Schreier domain iff each matrix in $\mathbb{M}_2(R)$ of zero determinant is non-full (see [3], Thm. 1 or [14], Lem. 1). Each pre-Schreier domain is a Π_2 domain by Theorem 1.4. Similarly, if $Pic(R)$ is trivial, then R is a Π_2 ring.

The following notions introduced by Bass (see [1]), Shchedryk (see [17]), and McGovern (see [13]) (respectively) will be often used in what follows.

Definition 1.5. Let $n \in \mathbb{N}$. Recall that R has:

(1) *stable range* n and we write $sr(R) = n$ if n is the smallest natural number with the property that each $(a_1, \ldots, a_n, b) \in Um(R^{n+1})$ is reducible, i.e., there exists $(r_1, \ldots, r_n) \in R^n$ such that $(a_1+br_1, \ldots, a_n+br_n) \in Um(R^n)$ (when there exists no $n \in \mathbb{N}$ such that $sr(R) = n$, then $sr(R) := \infty$ and the convention is $\infty > n$);

(2) *(fractional) stable range 1.5* and we write $fsr(R) = 1.5$ if for each $(a, b, c) \in Um(R^3)$ with $c \neq 0$ there exists $r \in R$ such that $(a + br, c) \in Um(R^2)$;

(3) *almost stable range 1* and we write $asr(R) = 1$ if for each ideal I of R not contained in $J(R)$, $sr(R/I) = 1$.

Stable range type of conditions on all suitable unimodular tuples of a ring date back at least to Kaplansky. For instance, the essence of [8], Thm. 5.2 can be formulated in the language of this paper as follows: each triangular matrix in $Um(\mathbb{M}_2(R))$ is simply extendable iff for each $(a, b, c) \in Um(R^3)$, there exists $(e, f) \in R^2$ such that $(ae, be + cf) \in Um(R^2)$. The general (nontriangular) form of this reformulation is: a ring R is an SE_2 ring iff for each $(a, b, c, d) \in Um(R^4)$, there exists $(e, f) \in R^2$ such that $(ae + cf, be + df) \in Um(R^2)$ (see Theorem 4.3).

We recall that if R has (Krull) dimension d, then $sr(R) \leq d + 1$ (see [1], Thm. 11.1 for the noetherian case and see [6], Cor. 2.3 for the general case). Also, if R is a finitely generated algebra over a finite field of dimension 2 or if R is a polynomial algebra in 2 indeterminates over a field that is algebraic over a finite field, then $sr(R) \leq 2$ (see [19], Cors. 17.3 and 17.4).

Each SE_2 ring is an E_2 ring, but we do not know when the converse is true. However, we show that there exist 2×2 matrices that are extendable but are not simply extendable (see Example 6.1). Moreover, we have (see Section 6):

Theorem 1.6. *If $sr(R) \leq 2$, then the extendable and simply extendable properties on a matrix in $\mathbb{M}_2(R)$ are equivalent (hence R is an SE_2 ring iff it is an E_2 ring).*

Based on Theorem 1.4, in Section 6 we prove the following theorem.

Theorem 1.7. *Let R be an integral domain of dimension 1. Then the following properties hold:*

(1) Each matrix in $Um(\mathbb{M}_2(R))$ with nonzero determinant is simply extendable.

(2) Each triangular matrix in $Um(\mathbb{M}_2(R))$ is simply extendable.

(3) The ring R is a Π_2 domain iff it is an SE_2 (or an E_2) domain and iff $Pic(R)$ is trivial.

(4) Assume R is a Dedekind domain. The ring R is a Π_2 domain iff it is a principal ideal domain (PID).

Recall that R is a *Hermite* ring in the sense of Kaplansky, if $RUm(R^2) = R^2$, equivalently if each 1×2 matrix with entries in R admits diagonal reduction. If R is a Hermite ring, then $sr(R) \in \{1, 2\}$ (see [15], Prop. 8(i); see also [21], Thm. 2.1.2) and a simple induction on $n \in \mathbb{N}$ gives that $RUm(R^n) = R^n$; thus $RUm(\mathbb{M}_2(R)) = \mathbb{M}_2(R)$. It follows that a Hermite ring R is an EDR iff each matrix in $Um(\mathbb{M}_2(R))$ admits diagonal reduction (equivalently, it is simply completable, see Theorem 4.3). From the last two sentences and Theorem 1.6 we conclude:

Corollary 1.8. *Let R be a Hermite ring. Then R is an EDR iff it is an E_2 ring and iff it is an SE_2 ring.*

For almost stable range 1 we have the following applications (see Section 6):

Corollary 1.9. *Assume that $asr(R) = 1$. Then the following properties hold:*

(1) Each triangular matrix in $Um(\mathbb{M}_2(R))$ is simply extendable.

(2) ***(McGovern)*** *If R is a Hermite ring, then R is an EDR.*

Corollary 1.9(2) was first obtained in [13], Thm. 3.7. A second proof of Corollary 1.9(2) is presented in Remark 7.2.

Part II studies determinant liftable 2×2 matrices that generalize simply extendable matrices[1] and proves that each $J_{2,1}$ domain introduced in [12] is an EDR. Part III has applications to Bézout rings (i.e., to rings whose finitely generated ideals are principal). Part IV contains universal and stability properties, complements and open problems. Parts I to IV split the manuscript https://arxiv.org/abs/2303.08413.

2. Basic terminology and properties

In what follows we will use without extra comments the following two basic properties. For $(a,b,c) \in R^3$ we have $(a,bc) \in Um(R^2)$ iff $(a,b),(a,c) \in Um(R^2)$. If $(a,b) \in Um(R^2)$ and $c \in R$, then a divides bc iff a divides c.

A ring R is called *pre-Schreier*, if every nonzero element $a \in R$ is *primal*, i.e., if a divides a product bc of elements of R, there exists $(d,e) \in R^2$ such that $a = de$, d divides b and e divides c. Pre-Schreier domains were introduced by Zafrullah in [22]. A pre-Schreier integrally closed domain was called a *Schreier* domain by Cohn in [2]. Every *GCD* domain (in particular, every Bézout domain) is Schreier (see [2], Thm. 2.4). Products of pre-Schreier domains and quotients of *PID*s are pre-Schreier rings.

In an integral domain, an irreducible element is primal iff it is a prime. Thus an integral domain that has irreducible elements which are not prime, such as each noetherian domain which is not a UFD, is not pre-Schreier.

The *inner rank* of an $m \times n$ matrix over a ring is defined as the least positive integer r such that it can be expressed as the product of an $m \times r$ matrix and an $r \times n$ matrix; over fields, this notion coincides with the usual notion of rank. A square matrix is called *full* if its inner rank equals its order, and *non-full* otherwise. A 2×2 matrix is non-full iff its inner rank is 1, (i.e., it has a column-row decomposition).

We consider the subsets of $Um(R^3)$:

$$T_3(R) := Um(R^2) \times (R \setminus \{0\}) \text{ and } J_3(R) := Um(R^2) \times \big(R \setminus J(R)\big).$$

Proposition 2.1. *We have $fsr(R) = 1.5$ iff for each $(a,b,c) \in T_3(R)$ there exists $r \in R$ such that $(a+br,c) \in Um(R^2)$.*

Proof. The 'only if' part is clear. To check the 'if' part, let $(a,b,c) \in Um(R^3)$ with $c \neq 0$ and let $(x,y,z) \in R^3$ be such that $ax + by + cz = 1$. Thus $(a, by + cz, c) \in T_3(R)$ and hence there exists $r \in R$ such that $(a + byr + czr, c) \in Um(R^2)$. This implies $(a + byr, c) \in Um(R^2)$, thus $fsr(R) = 1.5$. □

Proposition 2.2. *We have $asr(R) = 1$ iff for each $(a,b,c) \in J_3(R)$ there exists $r \in R$ such that $(a+br,c) \in Um(R^2)$.*

Proof. See [13], Thm. 3.6 for the 'only if' part. For the 'if' part, for I an ideal of R not contained in $J(R)$ we check that $sr(R/I) = 1$. If $(a,b) \in R^2$ is such that $(a+I, b+I) \in Um\big((R/I)^2\big)$, let $(d,e) \in R^2$ and $c \in I$ be such that $ad+be+c = 1$. If $c \notin J(R)$, then for $(f,g) := (c,c) \in \big(I \setminus J(R)\big) \times I$ we have $(a, be+g, f) \in J_3(R)$. If $c \in J(R)$, then $ad + be = 1 - c \in U(R)$, hence $(a, be) \in Um(R^2)$ and for $(f,g) \in \big(I \setminus J(R)\big) \times \{0\}$ we have $(a, be+g, f) \in J_3(R)$. If $r \in R$ is such that $\big(a+(be+g)r, f\big) \in Um(R^2)$, then $a+I+(b+I)(er+I) \in U(R/I)$; so $sr(R/I) = 1$. □

Corollary 2.3. ***(1)*** *If $sr(R) = 1$, then $fsr(R) = 1.5$.*

(2) *If $fsr(R) = 1.5$, then $asr(R) = 1$.*

(3) *If $asr(R) = 1$, then $sr(R) \leq 2$.*

[1] A matrix $A \in Um(\mathbb{M}_2(R))$ will be called determinant liftable if there exists $B \in Um(\mathbb{M}_2(R))$ congruent to A modulo $R\det(A)$ and $\det(B)$=0.

Proof. If $sr(R) = 1$, then for each $(a, b, c) \in T_3(R)$, there exists $r \in R$ such that $a + rb \in U(R)$, so $(a + rb, c) \in Um(R^2)$, hence $fsr(R) = 1.5$ by Proposition 2.1. So part (1) holds. As $J_3(R) \subseteq T_3(R)$, part (2) follows from Propositions 2.1 and 2.2.

To check part (3), it suffices to show that each $(a, b, c) \in Um(R^3)$ is reducible. If $b \notin J(R)$, then R/Rb has stable range 1 and $(a+Rb, c+Rb) \in Um\big((R/Rb)^2\big)$; hence there exists $r \in R$ such that $a+cr+Rb \in U(R/Rb)$ and thus for $(r_1, r_2) := (r, 0)$ we have $(a + cr_1, b + cr_2) \in Um(R^2)$. If $b \in J(R)$, then $(a, b, c) \in Um(R^3)$ implies that $(a, c), (a, b + c) \in Um(R^2)$ and thus for $(r_1, r_2) := (0, 1)$ we have $(a + cr_1, b + cr_2) \in Um(R^2)$. We conclude that (a, b, c) is reducible. □

Corollary 2.3(3) was first obtained in [13], Thm. 3.6.

We have the following 'classical' *units and unimodular interpretations.*

Proposition 2.4. ***(1)*** *For $n \in \mathbb{N}$, we have $sr(R) \leq n$ iff for each $b \in R$ the reduction modulo Rb map of sets $Um(R^n) \to Um\big((R/Rb)^n\big)$ is surjective.*

(2) *We have $fsr(R) = 1$ iff for all $(b, c) \in R^2$ with $c \neq 0$, the homomorphism $U(R/Rc) \to U\big(R/(Rb+Rc)\big)$ is surjective.*

(3) *We have $asr(R) = 1$ iff for each $(b, c) \in R^2$ with $c \notin J(R)$, the homomorphism $U(R/Rc) \to U\big(R/(Rb + Rc)\big)$ is surjective.*

Proof. Parts (1) and (2) follow from definitions. To check the 'if' part of (3), let $(a, b, c) \in J_3(R)$. Then $a+Rb+Rc$ is a unit of $R/(Rb+Rc)$ and thus is the image of a unit of R/Rc, which is of the form $a+br+Rc$ with $r \in R$. Hence $(a + br, c) \in Um(R^2)$. From this and Proposition 2.2 it follows that $asr(R) = 1$. To check the 'only if' part of (3), let $a + Rb + Rc \in U\big(R/(Rb + Rc)\big)$. Thus $(a + Rc, b + Rc) \in Um\big((R/Rc)^2\big)$. As $Rc \not\subseteq J(R)$ and we assume $asr(R) = 1$, it follows that $sr(R/Rc) = 1$, hence there exists $r \in R$ such that $a + br + Rc \in U(R/Rc)$ maps to $a + Rb + Rc \in U\big(R/(Rb + Rc)\big)$. Thus the homomorphism $U(R/Rc) \to U\big(R/(Rb + Rc)\big)$ is surjective. □

Corollary 2.5. *Assume that $sr(R) \leq 4$. If R is an E_2 (resp. SE_2) ring, then R/Ra is an E_2 (resp. SE_2) ring for all $a \in R$.*

Proof. Let $\bar{A} \in Um\big(\mathbb{M}_2(R/Ra)\big)$. As $sr(R) \leq 4$, there exists $A \in Um\big(\mathbb{M}_2(R)\big)$ whose reduction modulo Ra is $\bar{A}$ by Proposition 2.4(1) applied to $n = 4$. Let A^+ be an extension (resp. a simple extension) of A; its reduction modulo Ra is an extension (resp. a simple extension) of $\bar{A}$. So R/Ra is an E_2 (resp. SE_2) ring. □

Example 2.6. Let $(a, b, c) \in Um(R^3)$ with $(b, c) \in Um(R^2)$. Writing $a - 1 = be + cf$ with $e, f \in R$, for $r := -e$ we have $(a + br, c) \in Um(R^2)$.

Example 2.7. Suppose R is a semilocal ring. For each $b \in R$, the homomorphism $U(R) \to U(R/Rb)$ is surjective, hence $sr(R) = 1$ by Proposition 2.4(1).

Example 2.8. Let R be a noetherian domain of dimension 1. For $(b, c) \in R^2$ with $c \neq 0$, the rings R/Rc and $R/(Rb + Rc)$ are artinian, hence the homomorphism $U(R/Rc) \to U\big(R/(Rb + Rc)\big)$ is surjective. From this and Proposition 2.4(2), it follows that $fsr(R) = 1.5$; thus $sr(R) \leq 2$ (see Corollary 2.3(2) and (3)).

An argument similar to the one of Example 2.8 shows that each Bézout domain which is a filtered union of Dedekind domains has stable range 1.5.

Example 2.9. We have $sr(\mathbb{Z}) = 2$ and $fsr(\mathbb{Z}) = 1.5$. The ring $\mathbb{Z}[x]/(x^2)$ has almost stable range 1 but does not have stable range 1.5. For a finite field F and indeterminates X, Y, $sr(F[X,Y]) = sr(F[X]) = 2$; so $F[X,Y]$ does not have almost stable range 1. Hence, each possible converse of Corollary 2.3 does not hold.

3. Projective modules

For a projective R-module P of rank 1, let $[P] \in Pic(R)$ be its class in the Picard group. Let $Pic_2(R)$ be the subgroup of $Pic(R)$ generated by classes $[P]$ with P generated by 2 elements. Let $Pic_2(R)[2]$ be the subgroup of $Pic_2(R)$ generated by classes $[P]$ with $P \oplus P \cong R^2$ (so $2[P] = [R]$ and P is generated by 2 elements).

For $(\alpha_1, \ldots, \alpha_n) \in R^n$, let $Diag(\alpha_1, \ldots, \alpha_n) \in \mathbb{M}_n(R)$ be the diagonal matrix whose (i,i) entry is α_i for all $i \in \{1, \ldots, n\}$.

If $n \geq 2$ we do not have unit interpretations of the stable range n similar to Proposition 2.4 but this is replaced by standard projective modules considerations recalled here in the form required in the sequel. If P_1 and P_2 are two projective R-modules of rank 1 such that $P_1 \oplus P_2 \cong R^2$, then by taking determinants it follows that $P_1 \otimes_R P_2 \cong R$, hence $[P_1] = -[P_2] \in Pic(R)$; thus $P_1 \cong R$ iff $P_2 \cong R$.

Lemma 3.1. *Let $A \in Um\big(\mathbb{M}_2(R)\big)$. Let $\bar{R} := R/R\det(A)$ and let $\bar{A}$ be the reduction of A modulo $R\det(A)$. Then $\mathrm{Ker}_{\bar{A}}$, $\mathrm{Im}_{\bar{A}}$ and $\mathrm{Coker}_{\bar{A}}$ are projective $\bar{R}$-modules of rank 1 generated by two elements and we have $[\mathrm{Ker}_{\bar{A}}] = [\mathrm{Coker}_{\bar{A}}] = -[\mathrm{Im}_{\bar{A}}] \in Pic_2(R)$ (thus $\mathrm{Ker}_{\bar{A}} \cong \bar{R}$ iff $\mathrm{Im}_{\bar{A}} \cong \bar{R}$ and iff $\mathrm{Coker}_{\bar{A}} \cong \bar{R}$).*

Proof. As Coker_A is annihilated by $\det(A)$, we can view it as an $\bar{R}$-module isomorphic to $\mathrm{Coker}_{\bar{A}}$. Locally in the Zariski topology of the spectrum of $\bar{R}$, one of the entries of $\bar{A}$ is a unit and hence the matrices $\bar{A}$ and $Diag(1,0)$ are equivalent. Thus $\mathrm{Im}_{\bar{A}}$ and $\mathrm{Coker}_{\bar{A}}$ are projective $\bar{R}$-modules of rank 1 generated by two elements and we have two (split) short exact sequences of projective $\bar{R}$-modules $0 \to \mathrm{Im}_{\bar{A}} \to \bar{R}^2 \to \mathrm{Coker}_{\bar{A}} \to 0$ and $0 \to \mathrm{Ker}_{\bar{A}} \to \bar{R}^2 \to \mathrm{Im}_{\bar{A}} \to 0$. From the existence of $\bar{R}$-linear isomorphisms $\bar{R}^2 \cong \mathrm{Ker}_{\bar{A}} \oplus \mathrm{Im}_{\bar{A}} \cong \mathrm{Ker}_{\bar{A}} \oplus \mathrm{Coker}_{\bar{A}}$ it follows that $\mathrm{Ker}_{\bar{A}}$ and $\mathrm{Coker}_{\bar{A}}$ are isomorphic to the dual of $\mathrm{Im}_{\bar{A}}$ and the lemma follows. □

Next we exemplify the connection between projective R-modules and reducibility of $n+1$-tuples with entries in R.

Example 3.2. For $n \in \mathbb{N}$, let $(a_1, \ldots, a_n, b) \in Um(R^{n+1})$ with b a nonzero divisor. We consider short exact sequences of R-modules $0 \to R \xrightarrow{b} R \xrightarrow{\pi} R/Rb \to 0$ and $0 \to Q \to R^n \xrightarrow{f} R/Rb \to 0$ where π is the natural quotient map and f maps the elements of the standard basis of R^n to $a_1 + Rb, \ldots, a_n + Rb$ and $Q := Ker(f)$. If $Q^+ \to R^n$ and $Q^+ \to R$ define the pullback of f and π, then we have short exact sequences $0 \to Q \to Q^+ \to R$ and $0 \to R \to Q^+ \to R^n \to 0$ which imply that Q^+ is a free R-module of rank $n+1$ and Q is a projective R-module of rank n generated by $n+1$ elements; moreover, if $n = 1$, then $Q \cong R$.

We consider R-linear maps $g : R^n \to R$ such that $\pi \circ g = f$. Then $(a_1, \ldots, a_n, b)$ is reducible iff we can choose g to be surjective. Thus, if $sr(R) \leq n$, then we can choose g to be surjective.

If $A \in \mathbb{M}_n(R)$ is equivalent to $Diag(1, 1, \ldots, 1, b)$ and Im_A is the R-submodule Q of R^n (hence $Q \cong R^n$), then we can choose g to be surjective and hence $(a_1, \ldots, a_n, b)$ is reducible.

In this paragraph we assume that $(a_1, \ldots, a_n, b)$ is reducible and that g is chosen to be surjective. Then $Ker(g)$ is a projective R-module of rank $n-1$ generated by n elements, we have a short exact sequence $0 \to Ker(g) \to Q \to Rb \to 0$, and $0 \to Q \to R^n \xrightarrow{f} R/Rb \to 0$ is the direct sum of the two projective resolutions $0 \to R \xrightarrow{b} R \xrightarrow{\pi} R/Rb \to 0$ and $0 \to Ker(g) \to Ker(g) \to 0 \to 0$. If $n = 2$, then $Ker(g) \cong R$. Thus if $n = 2$ and the R-submodule Q of R^2 is Im_A for some $A \in \mathbb{M}_2(R)$, then A is equivalent to $Diag(1, b)$.

Similarly, if $n \geq 3$ and $Ker(g) \cong R^{n-1}$, then Q is a free R-module of rank n, so if the R-submodule Q of R^n is Im_A for some $A \in \mathbb{M}_n(R)$, then A is equivalent to $Diag(1, 1, \ldots, 1, b)$.

4. Criteria on extending 2×2 matrices

Lemma 4.1. *The following hold:*

(1) *A matrix $A \in \mathbb{M}_2(R)$ is extendable iff its reduction modulo $R\det(A)$ is simply extendable. Thus, if $\det(A) = 0$, then A is extendable iff it is simply extendable.*

(2) *For $M, N \in GL_2(R)$ and $Q \in SL_3(R)$ we have an identity*

$$\Theta\big(\sigma(M)Q\sigma(N)\big) = M\Theta(Q)N$$

and the $(3, 3)$ entries of Q and $\sigma(M)Q\sigma(N)$ are equivalent (thus one such entry is 0 iff the other entry is 0). Also, $\Theta(Q^T) = \Theta(Q)^T$.

(3) *The fact that a matrix $A \in \mathbb{M}_2(R)$ is (simply) extendable depends only on the equivalence class $[A] \in GL_2(R)\backslash\mathbb{M}_2(R)/GL_2(R)$. Moreover, A is (simply) extendable iff so is A^T. So $Im(\Theta)$ is stable under transposition and equivalence.*

Proof. To check the 'only if' part of part (1), let $A \in \mathbb{M}_2(R)$ be extendable, with $A^+ \in SL_3(R)$ an extension of it. If A_0^+ is obtained from A^+ by replacing the $(3, 3)$ entry with 0, then the reductions of A^+ and A_0^+ modulo $R\det(A)$ have the same determinant 1, and it follows that A modulo $R\det(A)$ is simply extendable. To check the 'if' part of (1), let $B \in \mathbb{M}_3(R)$ be such that $\Theta(B) = A$ and its reduction modulo $R\det(A)$ is a simple extension of the reduction of A modulo $R\det(A)$. Let $w \in R$ be such that $\det(B) = 1 + w\det(A)$. If $A^+ \in \mathbb{M}_3(R)$ is obtained from B by subtracting w from its $(3, 3)$ entry, then A^+ is an extension of A as $\det(A^+) = \det(B) - w\det(A) = 1$. Thus part (1) holds. Part (2) is a simple computation, while part (3) follows directly from part (2). □

Corollary 4.2. *We consider the following two statements on R.*

(1) *For each $a \in R$, R/Ra is a Π_2 ring.*

(2) *The ring R is an E_2 ring.*

Then $(1) \Rightarrow (2)$. If $sr(R) \leq 4$, then $1 \Leftrightarrow (2)$.

Proof. To prove that $(1) \Rightarrow (2)$, let $A \in Um\big(\mathbb{M}_2(R)\big)$. Its reduction modulo $R\det(A)$ has zero determinant and hence it is simply extendable as $R/R\det(A)$ is a Π_2 ring. From Lemma 4.1(1) it follows that A is extendable. Thus R is an E_2 ring. Hence $(1) \Leftrightarrow (2)$. If $sr(R) \leq 4$, then $(2) \Rightarrow (1)$ by Corollary 2.5, and hence $(1) \Leftrightarrow (2)$. □

Theorem 4.3. *For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ the following statements are equivalent:*

(1) *The matrix A is equivalent to the diagonal matrix $Diag\big(1, \det(A)\big)$.*

(2) *The matrix A is simply extendable.*

(3) *There exists $(e, f) \in R^2$ such that $(ae + cf, be + df) \in Um(R^2)$ (note that $(e, f) \in Um(R^2)$).*

(4) *There exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and the matrix $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is non-full.*

Proof. For $(e, f, s, t) \in R^4$ we have two identities for the determinant

$$\det \begin{bmatrix} a & b & f \\ c & d & -e \\ -t & s & 0 \end{bmatrix} = (be + df)t + (ae + cf)s = a(es) + b(et) + c(fs) + d(ft). \tag{I}$$

The equivalence (2) ⇔ (3) follows from the first identity. The implication (1) ⇒ (2) was checked before Proposition 1.3.

To show that (3) ⇒ (1) ∧ (4), let $(e, f) \in R^2$ be such that there exists $(s, t) \in Um(R^2)$ with $(ae + cf)s + (be + df)t = 1$. Then $(e, f), (s, t) \in Um(R^2)$ and thus there exists $M \in SL_2(R)$ whose first row is $[e\ f]$ and there exists $N \in SL_2(R)$ whose first column is $[s\ t]^T$. The matrix MAN has determinant $\det(A)$ and its $(1, 1)$ entry is 1, thus it is equivalent to the matrix $Diag\big(1, \det(A)\big)$ and therefore statement (1) holds by the transitivity of the equivalent relation. For $(x, y, z, w) := (es, et, fs, ft)$, then $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix} \begin{bmatrix} s & t \end{bmatrix}$ is non-full and from the second identity we get that $ax + by + cz + dw = 1$, thus statement (4) holds. If statement (4) holds, let $(e, f, s, t) \in R^4$ be such that $(x, y, z, w) = (es, et, fs, ft)$; as $ax + by + cz + dw = 1$, the determinant is 1 by the second identity, hence A is simply extendable. □

Remark 4.4. Referring to Theorem 4.3, as (2) ⇔ (3) and as a 2×2 matrix has a (simple) extension iff its transpose has it, it follows that A is simply extendable iff there exists $(e', f') \in R^2$ such that $(ae' + bf', ce' + df') \in Um(R^2)$.

Example 4.5. If $A^+ = \begin{bmatrix} a & b & f \\ c & d & -e \\ -t & s & 0 \end{bmatrix}$ is a simple extension of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$, then the characteristic polynomial χ_{A^+} of A^+ is of the form

$$x^3 - Tr(A)x^2 + \nu_{A^+}x - 1$$

(see Section 1 for $Tr(A) = a + d$ and $\nu_{A^+} = \det(A) + es + ft$). Thus the set of characteristic polynomials of simple extensions of A is in bijection to the subset $\nu(A) \subseteq R$ introduced in Section 1 and we have

$$\nu(A) = \{\det(A) + es + ft | (e, f, s, t) \in R^4,\ a(es) + b(et) + c(fs) + d(ft) = 1\}.$$

If $d \in R$, then $\nu\big(Diag(1, d)\big) = \{d + es + ft | (e, f, s, t) \in R^4,\ es + dft = 1\}$ is equal to $\{d + 1 - (d - 1)ft | (f, t) \in R^2\} = 2 + R(d - 1)$.

Concretely, if $R = \mathbb{Z}$, $\{(8 + 11k, -5 - 7k) | k \in \mathbb{Z}\}$ is the solution set of the equation $7es + 11ft = 1$ in es and ft, thus $\nu\big(Diag(7, 11)\big) = \{80 + 4k | k \in \mathbb{Z}\} = 4\mathbb{Z}$.

Corollary 4.6. ***(1)*** *The ring R is an SE_2 ring iff for each $(a, b, c, d) \in Um(R^4)$ statement (3) (or (4)) of Theorem 4.3 holds.*

(2) *Each semilocal ring is an SE_2 ring.*

Proof. Part (1) follows from definitions and Theorem 4.3. For part (2), based on part (1) it suffices to show that for each $(a, b, c, d) \in Um(R^4)$ there exists $(e, f) \in R^2$ such that $(ae + cf, be + df) \in Um(R^2)$. To prove this we can replace R by $R/J(R)$; thus R is a finite product of fields. By considering factors of R that are fields, we can assume that R is a field, in which case the existence of (e, f) is clear. □

Corollary 4.7. *For $(a, b, c, d) \in Um(R^4)$ the following properties hold:*

(1) *The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is extendable iff there exists $(e, f) \in R^2$ such that $(ae + cf, be + df, ad - bc) \in Um(R^3)$.*

(2) *The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is simply extendable iff there exists $(e, f) \in Um(R^2)$ such that $(ae + cf, be + df, ad - bc) \in Um(R^3)$*

Proof. Part (1) follows from Lemma 4.1(1) and Theorem 4.3. The 'only if' of part (2) follows from Theorem 4.3. We are left to prove that if $(e, f) \in Um(R^2)$ is such that $(ae+cf, be+df, ad-bc) \in Um(R^3)$, then A is simply extendable. Based on Theorem 4.3 it suffices to show that in fact we have $(ae+cf, be+df) \in Um(R^2)$. Let $I := R(ae+cf)+R(be+df)$ and $\mathfrak{m}$ a maximal ideal of R. If $ad-bc \in \mathfrak{m}$, then $(ae+cf, be+df, ad-bc) \in Um(R^3)$ implies that $I \not\subseteq \mathfrak{m}$. If $ad - bc \notin \mathfrak{m}$, then A modulo $\mathfrak{m}$ is invertible, hence $I \subseteq \mathfrak{m}$ iff $Re + Rf \subseteq \mathfrak{m}$; from this and $(e, f) \in Um(R^2)$ we infer that $I \not\subseteq \mathfrak{m}$. As I is not contained in any maximal ideal of R, it follows that $(ae + cf, be + df) \in Um(R^2)$. □

Corollary 4.8. *The following properties hold:*

(1) *The ring R is an E_2 ring iff for each $(a, b, c, d) \in Um(R^4)$ there exists $(e, f) \in R^2$ such that $(ae + cf, be + df, ad - bc) \in Um(R^3)$.*

(2) *The ring R is an SE_2 ring iff for each $(a, b, c, d) \in Um(R^4)$ there exists $(e, f) \in Um(R^2)$ such that $(ae + cf, be + df, ad - bc) \in Um(R^3)$.*

Proof. Both parts follow directly from definitions and the corresponding two parts of Corollary 4.7. □

Example 4.9. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$. In many simple cases one can easily prescribe $(e, f) \in R^2$ such that $(ae + cf, be + df) \in Um(R^2)$ and hence conclude that A is simply extendable. We include four such cases as follows.

(1) If $\{a, b, c, d\} \cap U(R) \neq \emptyset$, then we can take $(e, f) \in R^2$ such that $\{e, f\} = \{0, 1\}$. E.g., if $a \in U(R)$, then $\begin{bmatrix} a & b & 0 \\ c & d & -1 \\ 0 & a^{-1} & 0 \end{bmatrix}$ is a simple extension of A.

(2) If $\{(a, b), (a, c), (b, d), (c, d)\} \cap Um(R^2) \neq \emptyset$, then we can take $(e, f, e', f') \in R^4$ such that $1 \in \{ae + cf, be + df, ae' + bf', ce' + df'\}$. E.g., if $(a, b) \in Um(R^2)$ and $s, t \in R$ are such that $as + bt = 1$, then $\begin{bmatrix} a & b & 0 \\ c & d & -1 \\ -t & s & 0 \end{bmatrix}$ is a simple extension of A.

(3) If at least two of the entries a, b, c and d are in $J(R)$ (e.g., they are 0), then either $\{(a, b), (a, c), (b, d), (c, d)\} \cap Um(R^2) \neq \emptyset$ and part (2) applies or $(b, c) \in J(R)^2$ or $(a, d) \in J(R)^2$. The case $(a, d) \in J(R)^2$ is entirely similar to the case $(b, c) \in J(R)^2$, hence we detail here the case when $(b, c) \in J(R)^2$. As $(b, c) \in J(R)^2$, $(a, d) \in Um(R^2)$; let $(e, f) \in R^2$ be such that $1 = ae + df$. Then $\big(ae + bf + J(R), ce + df + J(R)\big) \in Um\big((R/J(R))^2\big)$, hence $(ae + bf, ce + df) \in Um(R^2)$; e.g., if $b = c = 0$, then $\begin{bmatrix} a & 0 & f \\ 0 & d & -e \\ -1 & 1 & 0 \end{bmatrix}$ is a simple extension of A.

(4) If $a, b, c, d, f, q \in R$ are such that $aq + df = 1$, then a simple extension of $\begin{bmatrix} a & ab \\ ac & d \end{bmatrix}$ is $\begin{bmatrix} a & ab & f \\ ac & d & -q + cf(1-b) \\ -1 & 1-b & 0 \end{bmatrix}$.

In three of these examples, the $(2,3)$ entry of the simple extensions is -1, i.e., we can choose $e = 1$. Such extensions relate to stable ranges 1 and 1.5 as follows.

Corollary 4.10. *The following properties hold:*

(1) *We have $sr(R) = 1$ iff each upper triangular matrix $A \in Um(\mathbb{M}_2(R))$ has a simple extension whose $(2,3)$ entry is -1.*

(2) *We have $fsr(R) = 1.5$ iff each upper triangular matrix $A \in Um(\mathbb{M}_2(R))$ with nonzero $(1,1)$ entry has a simple extension whose $(2,3)$ entry is -1.*

(3) *We have $asr(R) = 1$ iff each upper triangular matrix $A \in Um(\mathbb{M}_2(R))$ with $(1,1)$ entry not in $J(R)$ has a simple extension whose $(2,3)$ entry is -1.*

Proof. Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$. If $a = 0$, then $(b, d) \in Um(R^2)$ and from Equation (I) it follows that A has a simple extension with the $(2,3)$ entry -1 iff there exists $(f, t) \in R^2$ such that $bt + dft = 1$ and hence iff there exists $f \in R$ such that $b + df \in U(R)$. Thus all these matrices A with $a = 0$ have a simple extension with the $(2,3)$ entry -1 iff $sr(R) = 1$.

Similarly, if $a \neq 0$ (resp. $a \notin J(R)$), then from Equation (I) it follows that A has a simple extension with the $(2,3)$ entry -1 iff there exists $(e, f, t) \in R^3$ such that $ae + bt + dft = 1$ and hence iff there exists $f \in R$ such that $(b + df, a) \in Um(R^2)$. From the definition of stable range 1.5 (resp. almost stable range 1) applied to $(b, d, a) \in Um(R^3)$ it follows that all these matrices with $a \neq 0$ (resp. $a \notin J(R)$) have a simple extension with the $(2,3)$ entry -1 iff $fsr(R) = 1.5$ (resp. $asr(R) = 1$). □

5. Proof of Theorem 1.4

Proposition 5.1. *Let $A \in Um(\mathbb{M}_2(R))$. Then the following properties hold:*

(1) *Assume that $\det(A) = 0$. Then A is simply extendable iff one of the three R-modules Im_A, Ker_A and Coker_A is isomorphic to R and iff A is non-full.*

(2) *The matrix A is extendable iff its reduction modulo $R\det(A)$ is non-full.*

Proof. We prove part (1) using three (circular) implications; hence $\det(A) = 0$.

If A is non-full, we write $A = \begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$. As $A \in Um(\mathbb{M}_2(R))$, it follows that $(l, m), (o, q) \in Um(R^2)$; let $(e, f), (s, t) \in R^2$ be such that $el + fm = so + tq = 1$. Thus $\begin{bmatrix} e & f \end{bmatrix} A = \begin{bmatrix} o & q \end{bmatrix}$. Hence A is simply extendable by Theorem 4.3, a simple extension of it being $\begin{bmatrix} lo & lq & f \\ mo & mq & -e \\ -t & s & 0 \end{bmatrix}$.

If A is simply extendable, it admits diagonal reduction by Theorem 4.3. Thus, A, being unimodular with $\det(A) = 0$, is equivalent to $Diag(1, 0)$, so $\mathrm{Im}(A) \cong R$.

If one of the three R-modules Im_A, Ker_A and Coker_A is isomorphic to R, then all of them are isomorphic to R by Lemma 3.1. As $\mathrm{Im}_A \cong R$, the R-linear $L_A : R^2 \to R^2$ is a composite R-linear map $R^2 \to R \to R^2$, hence A is non-full.

Thus part (1) holds. Part (2) follows from part (1) and Lemma 4.1(1). □

We are now ready to prove Theorem 1.4. If $A \in Um\big(\mathbb{M}_2(R)\big)$ has zero determinant, then Ker_A and Im_A are projective R-modules of rank 1 dual to each other and $\mathrm{Ker}_A \oplus \mathrm{Im}_A \cong R^2$ by Lemma 3.1. Hence (4) ⇒ (3). The equivalences (1) ⇔ (2) ⇔ (3) follow from Proposition 5.1(1).

We show that (2) ⇒ (4). Each projective R-module P of rank 1 generated by 2 elements is isomorphic to Im_A for some idempotent $A \in \mathbb{M}_2(R)$ of rank 1 and hence unimodular of zero determinant. Assuming that (2) holds, we write $A = \begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$ with $(l,m),(o,q) \in Um(R^2)$, hence L_A is the composite of a surjective R-linear map $R^2 \to R$ and an injective R-linear map $R \to R^2$. Thus $P \cong \mathrm{Im}_A \cong R$, hence (2) ⇒ (4). Thus Theorem 1.4 holds.

Example 5.2. For $A \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(A) = 0$, we consider the R-submodule $K := R\begin{bmatrix} -b \\ a \end{bmatrix} + R\begin{bmatrix} -d \\ c \end{bmatrix}$ of Ker_A. For a maximal ideal $\mathfrak{m}$ of R, $K \not\subseteq \mathfrak{m}\mathbb{M}_2(R)$, hence $K \not\subseteq \mathfrak{m}\mathrm{Ker}_A$. Thus $\mathrm{Ker}_A = K$. For $(e,f) \in R^2$, we have $(ae+cf, be+df) \in Um(R^2)$ iff Ker_A is free having $e\begin{bmatrix} -b \\ a \end{bmatrix} + f\begin{bmatrix} -d \\ c \end{bmatrix}$ as a generator.

Example 5.3. Let $n \in \mathbb{N}$ and let $x_1, \ldots, x_n$ be indeterminates. Let $k \in \mathbb{N}$. Let $q := 4k+1$, $r := 2k+1$ and $\theta := i\sqrt{q} \in \mathbb{C}$. We check that $\mathbb{Z}[x_1, \ldots, x_n]$ *is a* Π_2 *ring which is not an* E_2 *ring.* As $\mathbb{Z}[x_1, \ldots, x_n]$ is a UFD, it is also a Schreier domain and hence a Π_2 ring (see Section 1). Thus it suffices to show that the matrix

$$A := \begin{bmatrix} r & 1-x_1 \\ 1+x_1 & 2 \end{bmatrix} \in Um\big(\mathbb{M}_2(\mathbb{Z}[x_1, \ldots, x_n])\big)$$

is not extendable. As $\det(A) = x_1^2 + q$, based on Lemma 4.1(1), it suffices to show that the image $B := \begin{bmatrix} r & 1-\theta \\ 1+\theta & 2 \end{bmatrix} \in Um\big(\mathbb{M}_2(\mathbb{Z}[\theta])\big)$ of A, via the composite homomorphism $\mathbb{Z}[x_1, \ldots, x_n] \to \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2+q) \to \mathbb{Z}[\theta]$ that maps x_1 to θ and $x_2, \ldots, x_n$ to 0, is not simply extendable. An argument on norms shows that the element $2 \in \mathbb{Z}[\theta]$ is irreducible, i.e., $2 = u(2u^{-1})$ with $u \in U(\mathbb{Z}[\theta])$ are its only product decompositions. So, as $2u^{-1}$ divides neither $1-\theta$ nor $1+\theta$, B is full. So B is not simply extendable by Proposition 5.1(1) and the integral domain $\mathbb{Z}[\theta]$ is not a Π_2 ring. If $4k+1$ is square free, then $\mathbb{Z}[\theta]$ is a Dedekind domain but not a *PID*.

Remark 5.4. Statements (1) to (4) of Theorem 4.3 are stable under similarity (inner automorphisms of the R-algebra $\mathbb{M}_2(R)$) but in general they are not stable under all R-algebra automorphisms of $\mathbb{M}_2(R)$. To check this, let R be such that there exists an R-module P such that $P \oplus P = R^2$ but $P \not\cong R$; so $[P] \in Pic_2(R)[2]$. The idempotent A of $\mathbb{M}_2(R)$ which is a projection of R^2 on the first copy of P along the second copy of P satisfies $\mathrm{Im}_A = P$ and $\det(A) = 0$, so it is not simply extendable by Theorem 1.4 but its image under the R-algebra automorphism $\mathbb{M}_2(R) = End_R(P \oplus P) \cong \mathbb{M}_2(R)$ defined by $End_R(P) \cong R$ maps A to $Diag(1,0)$. E.g., if R is a Dedekind domain with $Pic(R) \cong \mathbb{Z}/2\mathbb{Z}$ (such as $\mathbb{Z}[\sqrt{-5}]$), then we can take P to be a maximal ideal of R with nontrivial class in $Pic(R)$.

6. Proofs of Theorems 1.6 and 1.7 and Corollary 1.9

We show that if $sr(R) \leq 2$, then each extendable $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ is simply extendable. Let $(e', f') \in R^2$ be such that $(ae'+cf', be'+df', ad-bc) \in Um(R^3)$ by Corollary 4.7(1). Thus $(e', f', ad-bc) \in Um(R^3)$. As $sr(R) \leq 2$, there exists $(r_1, r_2) \in R^2$ such that

$$(e, f) := \big(e' + (ad - bc)r_1, f' + (ad - bc)r_2\big) \in Um(R^2).$$

As the ideals of R generated by $ae+cf, be+df, ad-bc$ and by $ae'+cf', be'+df', ad-bc$ are equal, we have $(ae+cf, be+df, ad-bc) \in Um(R^3)$. Hence A is simply extendable by Corollary 4.7(2). Thus Theorem 1.6 holds.

Example 6.1. Let R be an integral domain such that $sr(R) = 3$ and each projective R-module P with $P \oplus R \cong R^3$ is free. E.g., if κ is a subfield of $\mathbb{R}$, then $sr(\kappa[x_1, x_2]) = 3$ by [18], Thm. 8 and Seshadri proved that all finitely generated projective modules over it are free (see [16], Thm.; see also [9], Ch. XXI, Sect. 3, Thm. 3.5 for Quillen–Suslin Theorem). Let $(a_1, a_2, b) \in Um(R^3)$ be not reducible; thus $b \neq 0$. We have projective resolutions $0 \to Rb \to R \to R/Rb \to 0$ and $0 \to P \xrightarrow{g} R^2 \xrightarrow{f} R/Rb \to 0$, where the R-linear map f maps the elements of the standard basis of R^2 to $a_1 + Rb$ and $a_2 + Rb$, $P = \mathrm{Ker}(f)$ and g is the inclusion. As P is of the type mentioned (see Example 3.2), we identify $P = R^2$. Let $A \in \mathbb{M}_2(R)$ be such that $L_A = g : R^2 = P \to R^2$; we have $A \in Um\big(\mathbb{M}_2(R)\big)$ and $R\det(A) = Rb$. Let $\bar{A}$ be the reduction of A modulo $R\det(A)$. The $R/R\det(A)$-module Coker_A is isomorphic to $R/R\det(A)$ and to $\mathrm{Coker}_{\bar{A}}$. Thus $\bar{A}$ is simply extendable by Proposition 5.1(1). Hence A is extendable by Lemma 4.1(1). But A is not simply extendable: if it were, then it would be equivalent to $Diag\big(1, \det(A)\big)$ (see Theorem 4.3) and it would follow from Example 3.2 that $(a_1, a_2, b) \in Um(R^3)$ is reducible, a contradiction.

To prove Theorem 1.7, let R be an integral domain of dimension 1. To prove part (1), let $A \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(A) \neq 0$. The ring $\bar{R} := R/\det(A)R$ has dimension 0 and hence $Pic(\bar{R})$ is trivial. From this and Theorem 1.4 it follows that $\bar{R}$ is a Π_2 ring and therefore the reduction $\bar{A}$ of A modulo $R\det(A)$ is simply extendable. From Theorem 1.4 it follows that $\mathrm{Im}_{\bar{A}} \cong R/R\det(A)$. As R is an integral domain, we have two projective resolutions $0 \to R\det(A) \to R \to \mathrm{Im}_{\bar{A}} \to 0$ and $0 \to \det(A)R^2 \to \mathrm{Im}_A \to \mathrm{Im}_{\bar{A}} \to 0$ of R-modules. From this and Example 3.2 applied to $b = \det(A)$ (recall that $sr(R) \leq 2$), it follows that A is equivalent to $Diag\big(1, \det(A)\big)$ and hence is simply extendable (see Theorem 4.3).

Part (2) holds as triangular matrices in $Um\big(\mathbb{M}_2(R)\big)$ have either two 0 entries or nonzero determinants and thus are simply extendable by Example 4.9(3) or part (1). The first 'iff' of part (3) follows from part (1) and definitions. The isomorphism classes of projective R-modules of rank 1 are the isomorphisms classes of nonzero ideals of R which locally in the Zariski topology are principal; as for each $a \in R \setminus \{0\}$, $\dim(R/Ra) = 0$ and hence $Pic(R/Ra)$ is trivial, all such nonzero ideals are generated by 2 elements. From this and Theorem 1.4 it follows that R is a Π_2 domain iff $Pic(R)$ is trivial. Hence part (3) holds. As *PID*s are precisely Dedekind domains with trivial Picard groups, part (4) follows from the second 'iff' of part (3). Thus Theorem 1.7 holds.

Remark 6.2. The existence of an extension of a matrix in $Um\big(\mathbb{M}_2(R)\big)$ does not depend only on *the set* of its entries. This is so as, referring to Example 5.3, the matrix $\begin{bmatrix} 1+\theta & 1-\theta \\ r & 2 \end{bmatrix} \in Um\big(\mathbb{M}_2(\mathbb{Z}[\theta])\big)$ has the same entries as B, has nonzero determinant, and *it is* extendable (see Theorem 1.7(1)).

To prove Corollary 1.9, we assume that $asr(R) = 1$. Each matrix $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ is simply extendable by Example 4.9(3) if $a \in J(R)$ and by Corollary 4.10(3) if $a \notin J(R)$. So part (1) holds. To prove part (2), as R is also a Hermite ring, each $A \in Um\big(\mathbb{M}_2(R)\big)$ is equivalent to a triangular matrix and hence from part (1) and Lemma 4.1(3) it follows that A is simply extendable and thus admits diagonal reduction by Theorem 4.3. As $\mathbb{M}_2(R) = RUm\big(\mathbb{M}_2(R)\big)$ it follows that each matrix in $\mathbb{M}_2(R)$ admits diagonal reduction, hence it is equivalent to a diagonal matrix. Thus R is an EDR. Hence Corollary 1.9 holds.

7. Explicit computations for integral domains

Let $A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$ be such that we can write $a = ga'$, $c = gc'$, $b = hb'$, $d = hd'$ with $a', b', c', d', g, h \in R$ and $(a', c'), (b', d') \in Um(R^2)$. We have $(g, h) \in Um(R^2)$. Let $e', f' \in R$ be such that $a'e' + c'f' = 1$. Let $l := b'c' - a'd' \in R$ and $m := b'e' + d'f' \in R$; note that $\det(A) = -ghl$. As $[\, l \quad m \,] = [\, b' \quad d' \,] \begin{bmatrix} c' & e' \\ -a' & f' \end{bmatrix}$, the matrix $\begin{bmatrix} c' & e' \\ -a' & f' \end{bmatrix}$ has determinant 1, and, due to $(b', d') \in Um(R^2)$, it follows that $(l, m) \in Um(R^2)$.

Let $(e, f, w) \in R^3$ be such that $ae + cf = gw$. If R is an integral domain and $g \neq 0$, there exists $v \in R$ such that $(e, f) = (we' + c'v, wf' - a'v)$, hence

$$be + df = h(b'e + d'f) = h(b'we' + b'c'v + d'wf' - a'd'v) = hw(b'e' + d'f') + hv(b'c' - a'd')$$

is equal to $h(wm + vl)$. Thus $(ae + cf, be + df) = \big(gw, h(wm + vl)\big)$.

Proposition 7.1. *Let R be an integral domain. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$ be such that the above notation g, h, a', b', c', d' applies and let $(e', f', l, m) \in R^4$ be obtained as above. We assume $g \neq 0$.*

(1) *The matrix A is simply extendable iff there exists $(w, v) \in R^2$ such that $(g, wm + vl), (w, hvl) \in Um(R^2)$, in which case a simple extension of A is*

$$\begin{bmatrix} a & b & wf' - a'v \\ c & d & -we' - c'v \\ -t & s & 0 \end{bmatrix}$$

where $s, t \in R$ are such that $gws + h(wm + vl)t = 1$.

(2) *If the intersection $\{(g, l), (g, m), (h, l), (h, m)\} \cap Um(R^2)$ is nonempty (e.g., if $hlm = 0$), then A is simply extendable and $w, v \in R$ are given by formulas.*

Proof. There exists $(e, f) \in R^2$ such that $(ae + cf, be + df) \in Um(R^2)$ iff there exists $(w, v) \in R^2$ such that $\big(gw, h(wm + vl)\big) \in Um(R^2)$ (see above). We have $\big(gw, h(wm + vl)\big) \in Um(R^2)$ iff $\big(g, h(wm + vl)\big), \big(w, h(wm+vl)\big) \in Um(R^2)$. As $(g, h) \in Um(R^2)$, we have $\big(g, h(wm+vl)\big) \in Um(R^2)$ iff $(g, wm+vl) \in Um(R^2)$; moreover, $Rw + Rh(wm + vl) = Rw + Rhvl$. Thus $\big(gw, h(wm + vl)\big) \in Um(R^2)$ iff $(g, wm + vl), (w, hvl) \in Um(R^2)$. Based on the 'iff' statements of this paragraph and Theorem 4.3, it follows that part (1) holds.

To check part (2), we first notice that if $hlm = 0$, then the intersection is nonempty; e.g., if $h = 0$, then $g \in U(R)$ and hence $(g, l), (g, m) \in Um(R^2)$. Based on part (1) it suffices to show that in all four possible cases, we can choose $(w, v) \in R^2$ such that $(g, wm + vl), (w, hvl) \in Um(R^2)$.

If $(g, l) \in Um(R^2)$, for $(w, v) := (g, 1)$ we have $Rg+R(wm+vl) = Rg+Rl = R$. As $(g, h), (g, l) \in Um(R^2)$, also $(w, hvl) = (g, hl) \in Um(R^2)$.

If $(g, m) \in Um(R^2)$, for $(w, v) := (1, 0)$ we have $(g, wm + vl) = (g, m) \in Um(R^2)$ and $(w, hvl) = (1, 0) \in Um(R^2)$.

If $(h, m) \in Um(R^2)$, then $(hl, m) \in Um(R^2)$ and there exists $(w, v') \in R^2$ such that $wm + hv'l = 1$; so $Rw + Rhv'l = R$. For $v := hv'$ we have $wm + vl = 1$ and so $(g, wm + vl) \in Um(R^2)$ and $(w, hvl) = (w, h^2v'l) \in Um(R^2)$ as $(w, hv'l) \in Um(R^2)$.

If $(h, l) \in Um(R^2)$, let $(p, q) \in R^2$ be such that $1 = lp + mq$. For $w := hq + l$ and $v := hp - m$ we compute $wm + vl = h(lp + mq) + ml - ml = h$, so $(g, wm + vl) = (g, h) \in Um(R^2)$. Also, $Rw + Rh =$

$R(hq+l)+Rh = Rl+Rh = R$ and $Rw+Rvl$ contains $wm+vl = h$ and hence it contains $Rw+Rh = R$. Thus $(w, vl) \in Um(R^2)$. As $(w,h),(w,vl) \in Um(R^2)$ it follows that $(w, hvl) \in Um(R^2)$. □

Remark 7.2. We include a second proof of Corollary 1.9(2) for Bézout domains. Assume R is a Bézout domain with $asr(R) = 1$. Based on the equivalence of statements (1) and (2) of Theorem 4.3, it suffices to show that each matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ is simply extendable. As R is a Hermite domain, the notation of this section applies. As $(g,h) \in Um(R^2)$, by the symmetry between the pairs (a,c) and (b,d), we can assume that $g \notin J(R)$. As $(m,l,g) \in J_3(R)$ and $asr(R) = 1$, there exists $v \in R$ such that $(m+lv, g) \in Um(R^2)$. We take $w := 1$. Hence $(w, hvl) \in Um(R^2)$ and $(g, wm+lv) = (g, m+lv) \in Um(R^2)$. From Proposition 7.1(1) it follows that A is simply extendable.

Example 7.3. For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(\mathbb{Z})\big)$, its simple extensions $A^+_{(e,f,s,t)} = \begin{bmatrix} a & b & f \\ c & d & -e \\ -t & s & 0 \end{bmatrix}$ are parameterized by the set (see Equation (I))

$$\gamma_A := \{(e,f,s,t) \in \mathbb{Z}^4 | a(es) + b(et) + c(fs) + d(ft) = 1\}.$$

The below examples were (initially) exemplified using a code written for $R = \mathbb{Z}$ by the second author. We have $\nu_{A^+_{(e,f,s,t)}} = \chi'_{A^+_{(e,f,s,t)}}(0) = \det(A) + es + ft$.

(1) If $a = 0$ and $d = 1+b+c$, then we can take $A^+ = \begin{bmatrix} 0 & b & -1 \\ c & 1+b+c & -1 \\ 1 & 1 & 0 \end{bmatrix}$.

Concretely, suppose $(b,c) = (3,2)$; then $d = 6$, $\det(A) = -6$, $\gamma_A = \{(e,f,s,t) \in \mathbb{Z}^4 | 3et + 2fs + 6ft = 1\}$ and $\nu(A) = \{-6 + es + ft | (e,f,s,t) \in \gamma_A\}$.

To solve the equation $3et + 2fs + 6ft = 1$, let $w := et + 2ft$. We get $2fs + 3w = 1$ with general solution $fs = -1+3k$, $w = 1-2k$, where $k \in \mathbb{Z}$. The general solution of the equation $et+2ft = 1-2k$ is $et = 2k-1-2l$ and $ft = 1-2k+l$, where $l \in \mathbb{Z}$. Let $m := l-2k+1$. It follows that $ft = m$, $et = 1-2k-2m$, $fs = -1+3k$ and the only constraint is that $ft = m$ divides $etfs = (3k-1)(1-2k-2m)$, i.e., divides $(3k-1)(2k-1)$. As $es + ft = m + \dfrac{-6k^2+5k-1}{m} - 6k + 2$, it follows that

$$\nu(A) = \{-4 + m - 6k + \frac{-6k^2+5k-1}{m} | (m,k) \in \mathbb{Z}^2,\ m \text{ divides } -6k^2+5k-1\}.$$

(2) If $c = 0$ and $d = 1-a+b$, then we can take $A^+ = \begin{bmatrix} a & b & -1 \\ 0 & 1-a+b & -1 \\ 1 & 1 & 0 \end{bmatrix}$ (cf. Corollary 4.8(2): for simple extensions of upper triangular matrices with nonzero $(1,1)$ entries over rings R with $fsr(R) = 1.5$, we can choose $e = 1$).

Concretely, suppose $(a,b) = (6,-10)$; hence $d = 15$ and $\det(A) = -90$. Thus $\gamma_A = \{(e,f,s,t) \in \mathbb{Z}^4 | 6es - 10et - 15ft = 1\}$.

To solve the equation $6es - 10et - 15ft = 1$, let $w := 2et + 3ft$. We get $6es - 5w = 1$ with general solution $es = 1+5k$, $w = 1+6k$, where $k \in \mathbb{Z}$. Then $2et + 3ft = 1+6k$ has the general solution $et = -1-6k+3l$, $ft = 1+6k-2l$, where $l \in \mathbb{Z}$. Let $m := l-2k$. It follows that $es = 1+5k$, $et = -1+3m$, $ft = 1+2k-2m$ are subject to the only constraint that $et = -1+3m$ divides $esft = (1+5k)(1+2k-2m)$, i.e., it divides $(1+5k)(2k+m)$. As $es + ft = 2 + 7k - 2m$, it follows that

$$\nu(A)=\{-88+7k-2m|(m,k)\in\mathbb{Z}^2,\ -1+3m \text{ divides } (1+5k)(2+2k+m)\}.$$

For $m=0$ (resp. $m=1$) it follows that $\nu(A)\supseteq 3+7\mathbb{Z}$ (resp. $\nu(A)\supseteq 1+14\mathbb{Z}$).

The matrices $Diag(6,-15), A_0:=A\in\mathbb{M}_2(\mathbb{Z}[\frac{1}{21}])$ are similar and one checks that $\nu(A_0)=-90+\frac{1}{6}+\frac{7}{2}+2\mathbb{Z}[\frac{1}{21}]=-86-\frac{1}{3}+2\mathbb{Z}[\frac{1}{21}]$.

(3) If $A=\begin{bmatrix}15&6\\10&14\end{bmatrix}$, we can take $A^+=\begin{bmatrix}15&6&-2\\10&14&1\\-1&-1&0\end{bmatrix}$; indeed we have $\det(A^+)=1\cdot 15-1\cdot 6+2\cdot 10-2\cdot 14=1$. The entries of A use double products of the primes $2,3,5,7$. We have $\gamma_A=\{(e,f,s,t)\in\mathbb{Z}^4|15es+6et+10fs+14ft=1\}$, $\det(A)=150$, $\nu_{A^+}=149$, and $\nu(A)=\{150+es+ft|(e,f,s,t)\in\gamma_A\}$.

To solve the equation $15es+6et+10fs+14ft=1$, let $x:=5es+2et$ and $y:=5fs+7ft$, so we get $3x+2y=1$ with general solution $x=1+2k$, $y=-1-3k$, where $k\in\mathbb{Z}$. Then $5es+2et=1+2k$ has the general solution $es=1+2k+2l$, $et=-2(1+2k)-5l$, where $l\in\mathbb{Z}$, and $5fs+7ft=-1-3k$ has the general solution $fs=3(-1-3k)+7r$, $ft=-2(-1-3k)-5r$, where $r\in\mathbb{Z}$. Let $o:=k+l$, so $et=-2+4l-4o-5l=:q$. Thus $es=1+2o$, $et=q$, $l=-2-q-4o$, $k=2+q+5o$, therefore $fs=-3-18-9q-45o+7r$ and $ft=2+12+6q+30o-5r$. Let $p:=r-6o-q-3$. Thus $ft=-5p+q-1$ and $fs=-3o-2q+7p$. As $(es)(ft)=(et)(fs)$, we have an identity $(1+2o)(q-1-5p)=q(7p-3o-2q)$, which can be rewritten as $o(-2-10p+5q)=1+5p-q-2q^2+7pq$ and which for $q=2p$ becomes $-2o=6p^2+3p+1$, requiring p to be odd. For $q=2p$, $es+ft=2o+q-5p$ becomes $2o-3p=-6p^2-6p-1$. Thus

$$\nu(A)=\{150+2o+q-5p|(o,q,p)\in\mathbb{Z}^3,\ o(-2-10p+5q)=1+5p-q-2q^2+7pq\}$$

contains the set $\{150-6p^2-6p-1|p-1\in 2\mathbb{Z}\}$.

(4) If $A=\begin{bmatrix}30&42\\70&105\end{bmatrix}$, we can take $A^+=\begin{bmatrix}30&42&1\\70&105&3\\1&1&0\end{bmatrix}$; indeed we have $\det(A^+)=-3\cdot 30+3\cdot 42+1\cdot 70-1\cdot 105=1$. The entries of A use triple products of the primes $2,3,5,7$, hence they are not pairwise coprime.

CRediT authorship contribution statement

Grigore Călugăreanu: Writing – original draft. **Horia F. Pop:** Writing – original draft. **Adrian Vasiu:** Writing – original draft.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgement

The third author would like to thank SUNY Binghamton for good working conditions and Ofer Gabber for pointing out a glitch in one of the references. Competing interests: the authors declare none.

References

[1] H. Bass, K-theory and stable algebra, Inst. Hautes Études Sci. Publ. Math. 22 (1964) 5–60.
[2] P.M. Cohn, Bézout rings and their subrings, Math. Proc. Camb. Philos. Soc. 64 (1968) 251–264.

[3] G. Călugăreanu, H.F. Pop, On zero determinant matrices that are full, Math. Panon. N.S. 27/NS 1/ (2) (2021) 81–88.
[4] H.K. Farahat, W. Ledermann, Matrices with prescribed characteristic polynomials, Proc. Edinb. Math. Soc. 11 (1958/1959) 143–146.
[5] L. Gillman, M. Henriksen, Rings of continuous functions in which every finitely generated ideal is principal, Trans. Am. Math. Soc. 82 (1956) 366–391.
[6] R.C. Heitmann, Generating non-Noetherian modules efficiently, Mich. Math. J. 31 (2) (1984) 167–180.
[7] D. Hershkowitz, Existence of matrices with prescribed eigenvalues and entries, Linear Multilinear Algebra 14 (4) (1983) 315–342.
[8] I. Kaplansky, Elementary divisors and modules, Trans. Am. Math. Soc. 66 (1949) 464–491.
[9] S. Lang, Algebra, revised third edition, Grad. Texts in Math., vol. 211, Springer-Verlag, New York, 2002.
[10] M.D. Larsen, W.J. Lewis, T.S. Shores, Elementary divisor rings and finitely presented modules, Trans. Am. Math. Soc. 187 (1974) 231–248.
[11] J. Liu, D. Li, The generalized Serre problem over K-Hermite rings, J. Syst. Sci. Complex. 30 (2) (2017) 510–518.
[12] D. Lorenzini, Elementary divisor domains and Bézout domains, J. Algebra 371 (2012) 609–619.
[13] W.W. McGovern, Bézout rings with almost stable range 1, J. Pure Appl. Algebra 212 (2) (2008) 340–348.
[14] S. McAdam, D.E. Rush, Schreier rings, Bull. Lond. Math. Soc. 10 (1) (1978) 77–80.
[15] P. Menal, J. Moncasi, On regular rings with stable range 2, J. Pure Appl. Algebra 24 (1) (1982) 25–40.
[16] C.S. Seshadri, Triviality of vector bundles over the affine space K^2, Proc. Natl. Acad. Sci. USA 44 (1958) 456–458.
[17] V.P. Shchedryk, Some properties of primitive matrices over Bézout B-domain, Algebra Discrete Math. (2) (2005) 46–57.
[18] L.N. Vaseršteĭn, The stable range of rings and the dimension of topological spaces, Funkc. Anal. Prilozh. 5 (2) (1971) 17–27 (in Russian); English Translation: Funct. Anal. Appl. 5 (1971) 102–110.
[19] L.N. Vaseršteĭn, A.A. Suslin, Serre's problem on projective modules over polynomial rings, and algebraic K-theory, Izv. Akad. Nauk SSSR, Ser. Mat. 40 (5) (1976) 993–1054, 1199 (in Russian).
[20] R. Wiegand, S. Wiegand, Finitely generated modules over Bézout rings, Pac. J. Math. 58 (2) (1975) 655–664.
[21] B.V. Zabavsky, Diagonal Reduction of Matrices over Rings, Math. Stud. Monog. Ser., vol. 16, VNTL Publishers, L'viv, 2012, 251 pp.
[22] M. Zafrullah, On a property of pre-Schreier domains, Commun. Algebra 15 (9) (1987) 1895–1920.