MATRIX INVERTIBLE EXTENSIONS OVER COMMUTATIVE RINGS. PART II: DETERMINANT LIFTABILITY

GRIGORE CĂLUGĂREANU, HORIA F. POP, ADRIAN VASIU

ABSTRACT. A unimodular 2×2 matrix A with entries in a commutative ring R is called weakly determinant liftable if there exists a matrix B congruent to A modulo $R\det(A)$ and $\det(B) = 0$; if we can choose B to be unimodular, then A is called determinant liftable. If A is extendable to an invertible 3×3 matrix A^+, then A is weakly determinant liftable. If A is simple extendable (i.e., we can choose A^+ such that its $(3,3)$ entry is 0), then A is determinant liftable. We present necessary and/or sufficient criteria for A to be (weakly) determinant liftable and we use them to show that if R is a Π_2 ring in the sense of Part I (resp. is a pre-Schreier domain), then A is simply extendable (resp. extendable) iff it is determinant liftable (resp. weakly determinant liftable). As an application we show that each $J_{2,1}$ domain (as defined by Lorenzini) is an elementary divisor domain.

1. INTRODUCTION

Let R be a commutative ring with identity. For $n \in \mathbb{N} = \{1, 2, \ldots\}$, let $\mathbb{M}_n(R)$ be the R-algebra of $n \times n$ matrices with entries in R. We say that $B, C \in \mathbb{M}_n(R)$ are congruent modulo an ideal I of R if all entries of $B - C$ belong to I, i.e., $B - C \in \mathbb{M}_n(I)$. Let $SL_n(R) := \{M \in \mathbb{M}_n(R) | \det(M) = 1\}$. For a free R-module F, let $Um(F)$ be the set of *unimodular* elements of F, i.e., of elements $v \in F$ for which there exists an R-linear map $L : F \to R$ such that $L(v) = 1$.

In this paper we study a unimodular matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um(\mathbb{M}_2(R))$. Recall that A is called extendable if there exists $A^+ = \begin{bmatrix} a & b & f \\ c & d & -e \\ -t & s & v \end{bmatrix} \in SL_3(R)$ (see [2], Def. 1.1); we call A^+ an extension of A. If we can choose A^+ such that $v = 0$, then A is called simply extendable and A^+ is called a simple extension of A. Several characterizations of simply extendable matrices were proved in [2], Thm. 4.3. For instance, A is simply extendable iff there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and the matrix $N = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is non-full, i.e., it is a product $\begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$ with $(l, m, o, q) \in R^4$; thus $\det(N) = 0$. In practice, it is not easy to check if a unimodular matrix of zero determinant is non-full and hence one is led to weaken the non-full assumption and implicitly to identify classes of rings when the weakening is not an actual weakening. The natural weakening is to replace

2020 *Mathematics Subject Classification.* Primary: 15A83, 13G05, 19B10. Secondary: 13A05, 13F05, 13F25, 15B33, 16U10.

Key words and phrases. ring, matrix, projective module, unimodular.

the non-full assumption by the weaker $\det(N) = 0$ assumption. Modulo equivalent characterizations, we are led to introduce the following new notions.

Definition 1.1. *We say that $A \in Um\big(\mathbb{M}_2(R)\big)$ is weakly determinant liftable if there exists $B \in \mathbb{M}_2(R)$ congruent to A modulo $R\det(A)$ and $\det(B) = 0$. If there exists such a matrix B which is unimodular, then we remove the word 'weakly', i.e., we say that A is determinant liftable.*

If either A is invertible or $\det(A) = 0$, then A is determinant liftable. The (weakly) determinant liftability of A depends only on the equivalence class of A. The following characterizations of determinant liftability are proved in Section 3.

Theorem 1.2. *For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ the following statements are equivalent.*

(1) *The matrix A is determinant liftable.*

(2) *There exists $C \in Um\big(\mathbb{M}_2(R)\big)$ such that $A + \det(A)C \in Um\big(\mathbb{M}_2(R)\big)$ and* $\det(C) = \det\big(A + \det(A)C\big) = 0$.

(3) *There exists $(x, y, z, w) \in R^4$ such that $ax+by+cz+dw = 1$ and $xw-yz = 0$.*

(4) *There exists $C \in \mathbb{M}_2(R)$ such that* $\det(C) = \det\big(A + \det(A)C\big) = 0$.

Definition 1.1 is motivated by the following implications proved in Section 4.

Theorem 1.3. *For $A \in Um\big(\mathbb{M}_2(R)\big)$ the following properties hold.*

(1) *If A is simply extendable, then A is determinant liftable.*

(2) *If A is extendable, then A is weakly determinant liftable.*

Recall from [2], Def. 1.2(1) that R is called a Π_2 ring if each matrix in $Um\big(\mathbb{M}_2(R)\big)$ of zero determinant is extendable, equivalently it is simply extendable by [2], Lem. 4.1(1). If R is a Dedekind domain but not a PID, then it is not a Π_2 ring (see [2], Thm. 1.7(4)), so there exist matrices $A \in Um\big(\mathbb{M}_2(R)\big)$ of zero determinant that are not (simply) extendable, thus the converses of Theorem 1.3 do not hold in general.

We will use stable ranges and pre-Schreier domains as recalled in [2], Def. 1.5 and Sect. 2. Each pre-Schreier domain is a Π_2 ring (see [2], paragraph after Thm. 1.4). Recall from [2], Def. 1.2(3) that R is called an SE_2 ring if each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is simply extendable. The next two theorems are proved in Section 6.

Theorem 1.4. *The ring R is a Π_2 ring iff the simply extendable and determinant liftable properties on a matrix in $Um\big(\mathbb{M}_2(R)\big)$ are equivalent.*

From Theorem 1.4 and [2], Thm. 1.6 we get directly the following result.

Corollary 1.5. *If R is a Π_2 ring with $sr(R) \leq 2$, then the simply extendable, extendable and determinant liftable properties on a matrix in $Um\big(\mathbb{M}_2(R)\big)$ are equivalent.*

As SE_2 rings are Π_2 rings, from Theorem 1.4 we get directly the following result.

Corollary 1.6. *The ring R is an SE_2 ring iff it is an Π_2 ring with the property that each matrix in $Um\big(\mathbb{M}_2(R)\big)$ with nonzero determinant is determinant liftable.*

Theorem 1.7. *Assume R is such that each zero determinant matrix in $\mathbb{M}_2(R)$ is non-full (e.g., R is a product of pre-Schreier domains). Then a unimodular matrix $A \in Um(\mathbb{M}_2(R))$ is extendable iff it is weakly determinant liftable.*

From Theorems 1.3 and 1.7 we get directly the following result.

Corollary 1.8. *Assume R is such that $sr(R) \leq 2$ and each zero determinant matrix in $\mathbb{M}_2(R)$ is non-full (e.g., R is a product of pre-Schreier domains of stable range at most 2). Then the simply extendable, extendable, determinant liftable and weakly determinant liftable properties on a matrix in $Um\big(\mathbb{M}_2(R)\big)$ are equivalent.*

Example 1.9. Let R be a pre-Schreier domain such that $sr(R) = 3$ and there exists $A \in Um\big(\mathbb{M}_2(R)\big)$ which is extendable but not simply extendable (e.g., $R = K[X,Y]$ with K a subfield of $\mathbb{R}$, see [2], Ex. 6.1). Then A is weakly determinant liftable by Theorem 1.3(2) but it is not determinant liftable by Theorem 1.4. So the inequalities in Corollaries 1.5 and 1.8 are optimal.

Recall that R is a Hermite ring in the sense of Kaplansky if $R^2 = RUm(R^2)$. Lorenzini introduced 3 classes of rings that are 'between' elementary divisor rings and Hermite rings (see [8], Prop. 4.11). We define the first class, $J_{2,1}$, as follows.

Definition 1.10. *We say that R is:*

(1) *a $WJ_{2,1}$ ring if for each $(a,b,c,d) \in Um(R^4)$ and every $(\Psi, \Delta) \in R^2$, there exists $(x,y,z,w) \in R^4$ such that $ax + by + cx + dw = \Psi$ and $xw - yz = \Delta$;*

(2) *a $J_{2,1}$ ring if it is a Hermite ring and a $WJ_{2,1}$ ring.*

The above definition of a $J_{2,1}$ ring is equivalent to the one in [8], Def. 4.6 (see Proposition 7.1). In Section 7 we use Theorem 1.4 to solve a problem posed by Lorenzini (see [8], p. 618) and Fresnel (see [3], Subsect. 3.1) for the case of integral domains as follows.

Theorem 1.11. *Let R be a $WJ_{2,1}$ ring. Then the following properties hold.*

(1) *Each matrix $A \in Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable.*

(2) *Assume R is also a Hermite ring (i.e., R is a $J_{2,1}$ ring). Then R is an elementary divisor ring iff it is a Π_2 ring (thus the ring R is an elementary divisor domain iff it is an integral domain).*

Therefore the constructions for an arbitrary commutative ring performed in [8], Ex. 4.10 for $n = 2$ or Ex. 3.5 always produce rings which are either elementary divisor rings or are not Π_2 rings (in particular, the integral domains produced are elementary divisor domains).

For $J_{n,s}$ rings with $(n,s) \in \mathbb{N}^2$ see [8], Def. 4.6. By combining Theorem 1.11(2) with [8], Prop. 4.8 we get directly the following implication between these classes of rings.

Corollary 1.12. *Let R be a $J_{2,1}$ ring. If R is a Π_2 ring, then R is a $J_{n,1}$ ring for each integer $n > 1$.*

The R-algebras required in the proofs of above theorems are introduced in Section 2 and their main properties are presented in Theorem 2.1. Sections 5 and 9 prove criteria for weakly determinant liftability and determinant liftability via completions (respectively). Section 8 studies rings R for which all $A \in Um\big(\mathbb{M}_2(R)\big)$ are (weakly) determinant liftable. Part III will contain applications to Hermite rings.

2. Five Algebras

Let $N(R)$ and $Z(R)$ be the nilradical and the set of zero divisors of R (respectively). Let $Spec\ R$ be the spectrum of R. Let $Max\ R$ be the set of maximal prime ideals of R. For an element r of an R-algebra, (r) will be the principal ideal generated by it and $\bar{r}$ will be its reductions via surjective ring homomorphisms.

For $E \in \mathbb{M}_2(R)$ let $\mathrm{Tr}(E)$ be its trace, let $\mathrm{adj}(E) \in \mathbb{M}_2(R)$ be its adjugate, and let Ker_E be the kernel and Im_E be the image of the R-linear map $L_E : R^2 \to R^2$ defined by E. Let I_2 be the identity of $\mathbb{M}_2(R)$. Let X, Y, Z, W, V, U be indeterminates. Given A and elements x, y, z, w of an R-algebra, we define matrices

$$C = C_{(x,y,z,w)} := \begin{bmatrix} -w & z \\ y & -x \end{bmatrix},$$

$$D = D_{A,(x,y,z,w)} := I_2 + \mathrm{adj}(A)C = \begin{bmatrix} 1 - by - dw & bx + dz \\ ay + cw & 1 - ax - cz \end{bmatrix},$$

$$B = B_{A,(x,y,z,w)} := AD = A + \det(A)C = \begin{bmatrix} a - \det(A)w & b + \det(A)z \\ c + \det(A)y & d - \det(A)x \end{bmatrix}.$$

To $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ we attach five R-algebras. The R-algebra

$$\mathcal{U} = \mathcal{U}_A := R[X, Y, Z, W]/(1 - aX - bY - cZ - dW)$$

represents unimodular relations $ax + by + cz + dw = 1$ in R-algebras. The R-algebra

$$\mathcal{E} = \mathcal{E}_A := R[X, Y, Z, W, V]/\big(1 - aXW - bXZ - cYW - dYZ - (ad - bc)V\big),$$

represents extensions $\begin{bmatrix} a & b & y \\ c & d & -x \\ -z & w & v \end{bmatrix}$ of (images of) A in rings of 2×2 matrices with entries in R-algebras. Similarly, the R-algebra

$$\mathcal{S} = \mathcal{S}_A := R[X, Y, Z, W]/(1 - aXW - bXZ - cYW - dYZ) = \mathcal{E}/(\bar{V}),$$

represents simple extensions of (images of) A in rings of 2×2 matrices with entries in R-algebras. The polynomial

$$\Phi = \Phi_A(X, Y, Z, W) := 1 - aX - bY - cZ - dW + (ad - bc)(XW - YZ) \in R[X, Y, Z, W]$$

is a determinant with many decompositions of the form $e_{1,1}e_{2,2} - e_{1,2}e_{2,1}$, e.g., $\Phi = (1 - aX - cZ)(1 - bY - dW) - (aY + cW)(bX + dZ)$. Thus the R-algebra

$$\mathcal{W} = \mathcal{W}_A := R[X, Y, Z, W]/(\Phi)$$

represents 2×2 matrices $C = C_{(x,y,z,w)}$ with entries in R-algebras for which $\det(D) = 0$ and the R-algebra

$$\mathcal{Z} = \mathcal{Z}_A := R[X, Y, Z, W]/(1 - aX - bY - cZ - dW, XW - YZ) = \mathcal{W}/(\bar{X}\bar{W} - \bar{Y}\bar{Z}) = \mathcal{U}/(\bar{X}\bar{W} - \bar{Y}\bar{Z})$$

represents such matrices C for which $\det(C) = \det(D) = 0$; note that

$$\text{(I)} \qquad \det(D) = \Phi(x, y, z, w) \quad \text{and} \quad \det(B) = \det(A)\Phi(x, y, z, w).$$

If $c = 0$, then

$$\text{(II)} \qquad \begin{aligned} \mathcal{W} &= R[X, Y, Z, W]/\big((1 - aX)(1 - dW) - Y(b + adZ)\big) \\ &= R[X, Y, Z, W]/\big((1 - aX)(1 - bY - dW) - aY(bX + dZ)\big). \end{aligned}$$

The matrix A is extendable (resp. simply extendable) iff the R-algebra homomorphism $R \to \mathcal{E}$ (resp. $R \to \mathcal{S}$) has a retraction.

If $b = c$, then the R-algebras $\mathcal{U}$, $\mathcal{Z}$ and $\mathcal{W}$ have an involution defined by fixing X and W and by interchanging Y and Z, and we have

$$\mathcal{W}/(\bar{Y} - \bar{Z}) = R[X,Y,W]/\big((1 - aX - bY)(1 - bY - dW) - (bX + dY)(aY + bW)\big),$$

$$\mathcal{S}/(\bar{Y} - \bar{Z}) = R[X,Y,W]/\Big((1 - aXW) - Y\big(b(X + W) + dY\big)\Big).$$

We define an arrow diagram of R-algebra homomorphisms

(III)
$$\begin{array}{ccccc} \mathcal{W} & \longrightarrow & \mathcal{Z} & \longleftarrow & \mathcal{U} \\ & & \downarrow\rho & & \\ \mathcal{E} & \longrightarrow & \mathcal{S} & & \end{array}$$

as follows. The horizontal homomorphisms are epimorphisms defined by identifications $\mathcal{W}/(\bar{X}\bar{W} - \bar{Y}\bar{Z}) = \mathcal{Z} = \mathcal{U}/(\bar{X}\bar{W} - \bar{Y}\bar{Z})$ and $\mathcal{S} = \mathcal{E}/(\bar{V})$, and ρ is defined by mapping $\bar{X}$, $\bar{Y}$, $\bar{Z}$, $\bar{W}$ to $\bar{X}\bar{W}$, $\bar{X}\bar{Z}$, $\bar{Y}\bar{W}$, $\bar{Y}\bar{Z}$ (respectively). Equation (II) and Diagram (III) encode many applications in this Part II and the sequel Part III.

As $A \in Um\big(\mathbb{M}_2(R)\big)$, the R-linear map $M_A : R^4 \to R$ that maps $[x,y,z,w]^T$ to $ax + by + cz + dw$ is surjective. Thus

$$P = P_A := \mathrm{Ker}(M_A) = \{(x,y,z,w) \in R^4 | ax + by + cz + dw = 0\}$$

and its dual P^* are projective R-modules of rank 3 such that $P \oplus R \cong P^* \oplus R \cong R^4$. From [7], Ch. III, Sect. 6, Thm. 6.7 (1) it follows that $P \cong P^*$.

Theorem 2.1. *The following properties hold.*

(1) *For $f \in \{a,b,c,d\}$, the R_f-algebra $(\mathcal{U})_f$ is a polynomial R_f-algebra in 3 indeterminates. Also, the R-algebra $\mathcal{U}$ is isomorphic to the symmetric R-algebra of P (thus the homomorphism $R \to \mathcal{U}$ is smooth of relative dimension 3).*

(2) *The R-algebra $\mathcal{Z}$ is smooth of relative dimension 2.*

(3) *The R-algebra $\mathcal{E}$ is smooth of relative dimension 4.*

(4) *The R-algebra $(\mathcal{W})_{1-(ad-bc)(XW-YZ)}$ is smooth of relative dimension 3 (thus, if $ad - bc \in N(R)$, then the R-algebra $\mathcal{W}$ is smooth of relative dimension 3).*

(5) *The morphism of schemes Spec $\mathcal{S} \to$ Spec $\mathcal{Z}$ defined by ρ is a $\mathbb{G}_{m,\mathcal{Z}}$-torsor and hence it is smooth of relative dimension 1 (thus the R-algebra $\mathcal{S}$ is smooth of relative dimension 3).*

(6) *Assume $\det(A) = 0$. Then $\mathcal{W} = \mathcal{U}$ and for $f \in \{a,b,c,d\}$ the R_f-algebra $(\mathcal{Z})_f$ is a polynomial R_f-algebra in 2 indeterminates. Moreover, there exists a self-dual projective R-module Q of rank 2 such that the R-algebra $\mathcal{Z}$ is isomorphic to the symmetric R-algebra of Q and $Q \oplus R \cong P$ (thus $Q \oplus R^2 \cong R^4$).*

Proof. (1) As $A \in Um\big(\mathbb{M}_2(R)\big)$, we have *Spec* $R = \cup_{f\in\{a,b,c,d\}}$*Spec* R_f. To show that the R_f-algebra $(\mathcal{U})_f$ is a polynomial R_f-algebra we can assume that $f = a$ and by replacing (R, X) with $(R_f, a^{-1}X)$ we can assume that $a = 1$, in which case we have $\mathcal{U} \cong R[Y,Z,W]$. If $(a',b',c',d') \in R^4$ is such that $M_A(a',b',c',d') = 1$, then under the substitution $(X',Y',Z',W') := (a',b',c',d')+(X,Y,Z,W)$, the R-algebra $\mathcal{U} = R[X',Y',Z',W']/(aX' + bY' + cZ' + dW')$ is isomorphic to the symmetric R-algebra of the R-module $(X',Y',Z',W')/\big(aX'+bY'+cZ'+dW'+(X',Y',Z',W')^2\big)$ of quotient of ideals; as this R-module is isomorphic P^* and so to P, part (1) holds.

(2) It suffices to show that if $\mathfrak{n} \in Max\ (R[X,Y,Z,W])$ contains $XW-YZ$ and $1-aX-bY-cZ-dW$, then, denoting $\kappa := R[X,Y,Z,W]/\mathfrak{n}$, the κ-vector space

$$\kappa\delta X \oplus \kappa\delta Y \oplus \kappa\delta z \oplus \kappa\delta w/\big(\kappa\delta(aX+bY+cZ+dW)+\kappa\delta(XW-YZ)\big)$$

has dimension 2 (see [10], Exp. II, Thm. 4.10); here δ is the differential operator denoted in an unusual way in order to avoid confusion with the element $d \in R$. As $\delta(XW-YZ) = W\delta X - Z\delta Y - Y\delta Z + X\delta W$, to show this it suffices to show that the assumption that the reduction of the matrix

$$N_A := \begin{bmatrix} a & b & c & d \\ W & -Z & -Y & X \end{bmatrix}$$

modulo $\mathfrak{n}$ has rank ≤ 1, leads to a contradiction; as N_A modulo $\mathfrak{n}$ has unimodular rows, there exists $\alpha \in R[X,Y,Z,W]\setminus\mathfrak{n}$ such that $(a,b,c,d)+\alpha(W,-Z,-Y,X) \in \mathfrak{n}^4$. From this, as $XW-YZ, 1-aX-bY-cZ-dW \in \mathfrak{n}$, it follows that $2\alpha(XW-YZ)$ is congruent to both 0 and -1 modulo $\mathfrak{n}$, a contradiction. Thus part (2) holds.

(3) Based on [10], Exp. II, Thm. 4.10, it suffices to show that for

$$\Theta(X,Y,Z,W,V) := 1-aXW-bXZ-cYW-dYZ-(ad-bc)V \in R[X,Y,Z,W,V],$$

Θ and its partial derivatives Θ_X, Θ_Y, Θ_Z, Θ_W and Θ_V generate $R[X,Y,Z,W,V]$, but this follows directly from the identity $1 = \Theta - V\Theta_V - X\Theta_X - Y\Theta_Y$.

(4) Similar to (3), part (4) follows from the fact that $1-(ad-bc)(XW-YZ)$ is $\Phi - X\Phi_X - Y\Phi_Y - Z\Phi_Z - W\Phi_W$.

(5) The $\mathbb{G}_{m,\mathcal{Z}}$-action

$$Spec\ (\mathcal{S}[U,U^{-1}]) = Spec\ (\mathcal{Z}[U,U^{-1}]) \times_{Spec\ (\mathcal{Z})} Spec\ (\mathcal{S}) \to Spec\ (\mathcal{S})$$

is given by the R-algebra homomorphism $\mathcal{S} \to \mathcal{S}[U,U^{-1}]$ that maps $\bar{X}$, $\bar{Y}$, $\bar{Z}$, and $\bar{W}$ to $U\bar{X}$, $U\bar{Y}$, $U^{-1}\bar{Z}$, and $U^{-1}\bar{W}$ (respectively). The fact that it makes the morphism $Spec\ \mathcal{S} \to Spec\ \mathcal{Z}$ a $\mathbb{G}_{m,\mathcal{Z}}$-torsor is a standard exercise as the morphism ρ represents decompositions of $\begin{bmatrix} \bar{X} & \bar{Y} \\ \bar{W} & \bar{Z} \end{bmatrix} \in Um\big(\mathbb{M}_2(\mathcal{Z})\big)$ as a product $\begin{bmatrix} \bar{X}_{\mathcal{S}} \\ \bar{Y}_{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \bar{W}_{\mathcal{S}} & \bar{Z}_{\mathcal{S}} \end{bmatrix}$, the lower right index $\mathcal{S}$ emphasizing indeterminates for $\mathcal{S}$ (triviality over the open cover $\{Spec\ \mathcal{Z}_{\bar{X}}, Spec\ \mathcal{Z}_{\bar{Y}}, Spec\ \mathcal{Z}_{\bar{Z}}, Spec\ \mathcal{Z}_{\bar{W}}\}$ of $Spec\ \mathcal{Z}$); more precisely, say over $Spec\ \mathcal{Z}_{\bar{X}}$, as $\bar{X}$ is a unit of $\mathcal{Z}_{\bar{X}}$, given a product decomposition $\begin{bmatrix} \bar{X}_{\mathcal{S}} \\ \bar{Y}_{\mathcal{S}} \end{bmatrix} \begin{bmatrix} \bar{W}_{\mathcal{S}} & \bar{Z}_{\mathcal{S}} \end{bmatrix}$, any other product decomposition is of the form $\begin{bmatrix} U\bar{X}_{\mathcal{S}} \\ U\bar{Y}_{\mathcal{S}} \end{bmatrix} \begin{bmatrix} U^{-1}\bar{W}_{\mathcal{S}} & U^{-1}\bar{Z}_{\mathcal{S}} \end{bmatrix}$ for a uniquely determined unit U of $\mathcal{Z}_{\bar{X}}$.

(6) Clearly, $\mathcal{W} = \mathcal{U}$. For the polynomial R-algebra part we can assume that $f = a$. By eliminating $\bar{X} = a^{-1}(1-b\bar{Y}-c\bar{Z}-d\bar{W})$, we obtain an isomorphism

$$(\mathcal{Z})_a \cong R_a[Y,Z,W]/\big(YZ - a^{-1}W(1-bY-cZ-dW)\big),$$

which via the change of indeterminates $(Y_1,Z_1,W_1) := (aY+cW, aZ+bW, aW)$ is isomorphic, as $ad = bc$, to $R_a[Y_1,Z_1,W_1]/a^{-2}(Y_1Z_1-W_1) \cong R_a[Y_1,Z_1]$. From [1], Thm. 4.4 it follows that $\mathcal{Z}$ is isomorphic to the symmetric R-algebra of a projective R-module Q of rank 2. As $\mathcal{Z}$ is a symmetric R-algebra, there exists an R-algebra epimorphism (retraction) $\mathcal{Z} \to R$ and hence there exists a quadruple

$\zeta := (a', b', c', d') \in R^4$ such that $1 - M_A(\zeta) = 0 = a'd' - b'c'$. With the substitution $(X', Y', Z', W') := (a', b', c', d') + (X, Y, Z, W)$, we identify

$$\mathcal{Z} = R[X', Y', Z', W']/(aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W' - X'W' + Y'Z').$$

Endowing R^4 with the inner product $\langle (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\rangle := \sum_{i=1}^{4} x_i y_i$, we have $\langle \zeta, (d', -c', -b', a')\rangle = 0$ and $\langle \zeta, (a, b, c, d)\rangle = 1$. Therefore the R-submodule $W := R(a, b, c, d) + R(d', -c', -b', a')$ of R^4 is a direct sum (so $W \cong R^2$) and a direct summand (i.e., and R^4/W is a projective R-module of rank 2). Let

$$\mathcal{Z}' := R[X', Y', Z', W']/(aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W').$$

Note that $Spec\ R = \cup_{f \in \{a,b,c,d\}} Spec\ R_{ff'}$ and for $f \in \{a, d\}$ (resp. $f \in \{b, c\}$), the $R_{ff'}$-algebra $(\mathcal{Z}')_{ff'}$ is isomorphic to $R_{ff'}[Y', Z']$ (resp. $R_{ff'}[X', W']$). Let J and J' be the ideals of $\mathcal{Z}$ and $\mathcal{Z}'$ (respectively) generated by the images of X', Y', Z', W'. We view $\mathcal{Z}$ and $\mathcal{Z}'$ as augmented R-algebras (in the terminology of [1]) with the augmentations given by the natural identifications $\mathcal{Z}/J = \mathcal{Z}'/J' = R$ of R-algebras. The R-modules $Q := J/J^2$ and $Q' = J'/(J')^2$ are identified via the third isomorphism theorem with the following R-module quotient of ideals

$$(X', Y', Z', W')/\big((aX' + bY' + cZ' + dW', d'X' - c'Y' - b'Z' + a'W') + (X', Y', Z', W')^2\big)$$

of $R[X', Y', Z', W']$, thus are isomorphic to R^4/W. From [1], Cor. 4.3 and the above part on isomorphisms of localizations $(\mathcal{Z})_f$ and $(\mathcal{Z}')_{ff'}$ that involve indeterminates that are linear (not necessary homogeneous) polynomials in X', Y', Z', W', it follows that the augmented R-algebras $\mathcal{Z}$ and $\mathcal{Z}'$ are isomorphic to the symmetric R-algebras of Q and Q' (respectively) endowed with their natural augmentations, and so they are isomorphic. As W is a direct summand of R^4, there exists a short exact sequence $0 \to R \to P^* \to Q' \to 0$ of R-modules, so $Q \oplus R \cong P$. Thus $Q \oplus R^2 \cong P \oplus R^3 \cong R^4$. As the R-module Q is stable free of rank 2, it is self-dual by [7], Ch. III, Sect. 6, Thm. 6.8. □

Corollary 2.2. *Assume there exist two ideals* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *of* R *such that* $\mathfrak{i}_1 \cap \mathfrak{i}_2 = 0$ *and* $\det(A) \in \mathfrak{i}_2$. *Then for* $\mathcal{E} \in \{\mathcal{Z}, \mathcal{W}\}$, *each* R*-algebra homomorphism* $\mathcal{E} \to R/\mathfrak{i}_1$ *lifts to an* R*-algebra homomorphism* $\mathcal{E} \to R$.

Proof. Let $h_{1,2} : \mathcal{E} \to R/(\mathfrak{i}_1 + \mathfrak{i}_2)$ be induced by an R-algebra homomorphism $h_1 : \mathcal{E} \to R/\mathfrak{i}_1$. As A modulo $\mathfrak{i}_2$ has zero determinant, $\mathcal{E}/\mathfrak{i}_2\mathcal{E}$ is the symmetric algebra of a projective $R/\mathfrak{i}_2$-module Q_2 of rank 2 if $\mathcal{E} = \mathcal{Z}$ and of rank 3 if $\mathcal{E} = \mathcal{W}$ (see Theorem 2.1(1) and (6)). The R-algebra homomorphism $h_{1,2}$ is uniquely determined by an R-linear map $l_{1,2} : Q_2 \to R/(\mathfrak{i}_1 + \mathfrak{i}_2)$. If $l_2 : Q_2 \to R/\mathfrak{i}_2$ is an R-linear map that lifts $l_{1,2}$ and if $h_2 : \mathcal{E} \to R/\mathfrak{i}_2$ is the R-algebra homomorphism uniquely determined by l_2, then h_2 lifts $h_{1,2}$. As $\mathfrak{i}_1 \cap \mathfrak{i}_2 = 0$, we have a pullback diagram

$$\begin{array}{ccc} R & \longrightarrow & R/\mathfrak{i}_1 \\ \downarrow & & \downarrow \\ R/\mathfrak{i}_2 & \longrightarrow & R/(\mathfrak{i}_1 + \mathfrak{i}_2), \end{array}$$

so there exists a unique R-algebra homomorphism $\mathcal{E} \to R$ that lifts h_1 and h_2. □

Remark 2.3. **(1)** The fact that if the Picard group $Pic(R)$ is trivial then R is a Π_2 ring (see [2], paragraph after Thm. 1.4) also follows easily from Theorem 2.1(5) and (6) via the equivalence between $\mathbb{G}_m$-torsors and line bundles.

(2) For each $\mathfrak{m} \in Max\ R$, the image of A in $Um(\mathbb{M}_2(R_{\mathfrak{m}}))$ is simply extendable (for instance, see [2], Cor. 4.6(2)). Thus, if $ab = bc$ and $\mathcal{Z}$ is a polynomial R-algebra in 2 indeterminates, Quillen Patching Theorem (see [9], Thm. 1) implies that the $\mathbb{G}_{m,\mathcal{Z}}$-torsor of Theorem 2.1(5) is the pullback of a $\mathbb{G}_{m,R}$-torsor.

Example 2.4. Assume R is a Hermite ring. As P is stable free, it is free (see [11], Cor. 3.2); so $P \cong R^3$ and the R-algebra $\mathcal{U}$ is a polynomial R-algebra in 3 indeterminates. If $\det(A) = 0$, then similarly we argue that $Q \cong R^2$ and the R-algebra $\mathcal{Z}$ is a polynomial R-algebra in 2 indeterminates. We prove that the $\mathbb{G}_{m,\mathcal{Z}}$-torsor of Theorem 2.1(5) is the pullback of a $\mathbb{G}_{m,R}$-torsor. Due to the equivalence between $\mathbb{G}_m$-torsors and line bundles, it suffices to show that for a polynomial R-algebra R_1, the functorial homomorphism $Pic(R) \to Pic(R_1)$ is an isomorphism. To show this we can assume that R is reduced, i.e., $N(R) = 0$. For each $\mathfrak{p} \in Spec\ R$, the reduced local Hermite ring $R_{\mathfrak{p}}$ is a reduced valuation ring (see [5], Thms. 1 and 2), hence a valuation domain (this was already stated in [6], Sect. 10).[1] Thus R is a normal ring (i.e., all its localizations $R_{\mathfrak{p}}$ are integral domains that are integrally closed in their fields of fractions); hence it is a seminormal ring in the sense of [4]. From this and [4], Thm. 1.5 it follows that $Pic(R) \to Pic(R_1)$ is an isomorphism.

Example 2.5. Assume A is symmetric and $\det(A) = 0$. As $(a, b, c, d) \in Um(R^4)$, $b = c$, $ad = bc$, we have $(a, d) \in Um(R^2)$. Thus $Spec\ \mathcal{S} = Spec\ \mathcal{S}_a \cup Spec\ \mathcal{S}_d$. We have canonical R-algebra identifications $(\mathcal{S})_a = R_a[X_1, Y, Z, W_1]/(1 - X_1W_1)$ and $(\mathcal{S})_d = R_d[X, Y_1, Z_1, W]/(1 - Y_1Z_1)$, where $X_1 := aX + bY$, $W_1 := W + ba^{-1}Z$, $Y_1 := Y + bd^{-1}X$, and $Z_1 := dZ + bW$.

3. Proof of Theorem 1.2

For $\upsilon = (x, y, z, w) \in R^4$, let $C = C_\upsilon, B = B_{A,\upsilon}, D = D_{A,\upsilon} \in \mathbb{M}_2(R)$ be as in Section 2. As $\mathrm{Tr}(D) = 2 - ax - by - cz - dw$, from Equation (I) it follows that

$$1 - \det(A)\det(C) = \mathrm{Tr}(D) - \det(D). \tag{IV}$$

We first prove the following general lemma.

Lemma 3.1. *Let $G, H, E \in \mathbb{M}_2(R)$. Then the following properties hold.*

(1) *There exists a matrix $O \in \mathbb{M}_2(R)$ such that $H = G(I_2 + \mathrm{adj}(G)O)$ iff G and H are congruent modulo $R\det(G)$.*

(2) *If GE is unimodular, then G and E are unimodular.*

(3) *If G is unimodular and G and GE are congruent modulo $R\det(G)$, then GE is unimodular iff E is unimodular.*

Proof. As $G\mathrm{adj}(G) = \det(G)I_2$, for $O \in \mathbb{M}_2(R)$ we have $H = G + \det(G)O$ iff $H = G(I_2 + \mathrm{adj}(G)O)$. So part (1) holds. The only nontrivial impication of parts (2) and (3) is the 'if' of part (3). It suffices to show that the ideal $\mathfrak{h}$ of R generated by the entries of GE is not contained in any $\mathfrak{m} \in Max\ R$. This holds if $\det(G) \in \mathfrak{m}$ as G and GE are congruent modulo $R\det(G)$. If $\det(G) \notin \mathfrak{m}$, then G modulo $\mathfrak{m}$ is invertible, thus GE modulo $\mathfrak{m}$ is nonzero as this is so for E modulo $\mathfrak{m}$, so $\mathfrak{h} \not\subseteq \mathfrak{m}$. □

[1] Recall from [6], Sect. 10, Def. that a ring R is called a valuation ring if for each $(a, b) \in R^2$, either a divides b or b divides a, equivalently, if the ideals of R are totally ordered by set inclusion. Reduced valuation rings are integral domains. This is so as for a reduced local ring S which is not an integral domain there exists nonzero elements $a, b \in S$ such that $ab = 0$, thus the nilpotent ideal $Sa \cap Sb$ is 0, and it follows that the finitely generated ideal $Sa + Sb \cong Sa \oplus Sb$ is not principal.

To prove Theorem 1.2, we first remark that clearly (2) $\Rightarrow$ (1) $\wedge$ (4).

If $v = (x, y, z, w) \in R^4$ is such that $ax + by + cz + dw = 1$ and $xw - yz = 0$, then $\Phi(x, y, z, w) = 0$, $\det(C) = 0$, $C \in Um\big(\mathbb{M}_2(R)\big)$ and for $B = A + \det(A)C$ we have $\det(B) = 0$ (see Equation (I)); as $\mathrm{Tr}(D) - \det(D) = 1$ by Equation (IV), D is unimodular, so $B = AD$ is unimodular by Lemma 3.1(2), hence (3) $\Rightarrow$ (2) holds.

To show that (4) $\Rightarrow$ (3), let $v = (x, y, z, w) \in R^4$ be such that $C = C_v$ and assume that $\det(B) = \det(C) = 0$. Thus $xw - zy = 0$ and $\det(A)\Phi(x, y, z, w) = \det(B) = 0$ (by Equation (I)). If $\det(A) \notin Z(R)$, then $1 - ax - by - cz - dw = 0$, hence (4) $\Rightarrow$ (3).

In general, we have to show that the R-algebra homomorphism $R \to \mathcal{Z}$ has a retraction $\mathcal{Z} \to R$. By replacing R with a finitely generated $\mathbb{Z}$-subalgebra S of R such that $A, B \in Um\big(\mathbb{M}_2(S)\big)$ and $C \in \mathbb{M}_2(S)$, we can assume that R is noetherian. Thus the set of minimal prime ideals $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_j\}$ of R has a finite number of elements $j \in \mathbb{N}$. As the homomorphism $R \to \mathcal{Z}$ is smooth (see Theorem 2.1(2)), each R-algebra homomorphism $\mathcal{Z} \to R/N(R)$ lifts to an R-algebra epimorphism $\mathcal{Z} \to R$. So, by replacing R with $R/N(R)$, we can assume that $N(R) = \cap_{i=1}^{j} \mathfrak{p}_i = 0$. Let $\det(A)_i$ be the image of $\det(A)$ in $R/\mathfrak{p}_i$. Based on Theorem 2.1(6) we can assume that $\det(A) \neq 0$, and hence there exists an $i \in \{1, \ldots, j\}$ such that $\det(A) \notin \mathfrak{p}_i$, i.e., $\det(A)_i \neq 0$. We can assume that the minimal prime ideals are indexed such that there exists $j' \in \{1, \ldots, j\}$ for which $\det(A)_i \neq 0$ if $i \in \{1, \ldots, j'\}$ and $\det(A)_i = 0$ if $i \in \{j' + 1, \ldots, j\}$. If $\mathfrak{i}_1 := \cap_{i=1}^{j'} \mathfrak{p}_i$ and $\mathfrak{i}_2 := \cap_{i=j'+1}^{j} \mathfrak{p}_i$, we have $\mathfrak{i}_1 \cap \mathfrak{i}_2 = 0$ and $\det(A) \in \mathfrak{i}_2$. As $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$, from the prior paragraph it follows that there exists an R-algebra homomorphism $h_1 : \mathcal{Z} \to R/\mathfrak{i}_1$. From Corollary 2.2 we get that there exists a retraction $\mathcal{Z} \to R$ that lifts h_1. So (4) $\Rightarrow$ (3) holds.

We conclude that statements (2), (3) and (4) are equivalent and imply (1).

We prove that (1) $\Rightarrow$ (2). As (2) $\Leftrightarrow$ (3), as above we argue that it suffices to prove that (1) $\Rightarrow$ (2) when R is noetherian and $N(R) = 0$. Let the ideals $\mathfrak{i}_1$ and $\mathfrak{i}_2$ of R be as above. Let $B \in Um\big(\mathbb{M}_2(R)\big)$ be congruent to A modulo $R\det(A)$ and $\det(B) = 0$. Let $v = (x, y, z, w) \in R^4$ be such that $B = B_{A,v}$ (see Lemma 3.1(1)). With $C = C_v$ and $D = D_{A,v}$, as $B = AD \in Um\big(\mathbb{M}_2(R)\big)$ we have $D \in Um\big(\mathbb{M}_2(R)\big)$ (see Lemma 3.1(2)). As $\det(A) + \mathfrak{i}_1 \notin Z(R/\mathfrak{i}_1)$, from the identity $\det(B) = \det(A)\det(D) = 0$ and Equation (I) it follows that $\Phi(x, y, z, w) \in \mathfrak{i}_1$, hence there exists an R-algebra epimorphism $g_1 : \mathcal{W} \to R/\mathfrak{i}_1$ that maps the elements $\bar{X}, \bar{Y}, \bar{Z}, \bar{W}$ of $\mathcal{W}$ to $x + \mathfrak{i}_1, y + \mathfrak{i}_1, z + \mathfrak{i}_1, w + \mathfrak{i}_1$ (respectively). Let $g : \mathcal{W} \to R$ be an R-algebra homomorphism that lifts g_1 (see Corollary 2.2).

Let $v' = (x', y', z', w') := g^4(\bar{X}, \bar{Y}, \bar{Z}, \bar{Z}) \in R^4$. For the matrices $C' := C_{v'}$ and $D' := D_{A,v'}$ we have (see Equation (I)) $\det(D') = \Phi(x', y', z', w') = 0$ and C' and C are congruent modulo $\mathfrak{i}_1$. Hence D' and D are congruent modulo $\mathfrak{i}_1$. As D is unimodular, it follows that the ideal $\mathfrak{d}'$ of R generated by the entries of D' satisfies $\mathfrak{d}' + \mathfrak{i}_1 = R$ and thus $\mathfrak{d}'$ is not contained in any $\mathfrak{m} \in Max\ R$ with $\det(A) \notin \mathfrak{m}$. As $\mathrm{Tr}(D') - \det(D') = 1 - \det(A)\det(C') \in \mathfrak{d}'$ by Equation (IV), $\mathfrak{d}'$ is not contained in any maximal ideal which does not contain $1 - \det(A)\det(C')$. Hence $\mathfrak{d}'$ is not contained in any $\mathfrak{m} \in Max\ R$, thus $\mathfrak{d}' = R$, i.e., D' is unimodular. From Lemma 3.1(2) it follows that $B' := AD'$ is unimodular.

By replacing the triple (C, B, D) with (C', B', D'), we can assume that $\det(D) = 0$. As $D = I_2 + \mathrm{adj}(A)C$ has zero determinant, it follows that $C \in Um\big(\mathbb{M}_2(R)\big)$.

To complete the proof that (1) $\Rightarrow$ (2), it suffices to show that we can replace C by a matrix $C_1 \in Um\big(\mathbb{M}_2(R)\big)$ with $\det(C_1) = 0$ and such that for $D_1 := I_2 + \mathrm{adj}(A)C_1$ we have $\det(D_1) = 0$ and $D_1 \in Um\big(\mathbb{M}_2(R)\big)$: so $B_1 := AD_1$ is congruent to A

modulo $R\det(A)$ and unimodular by Lemma 3.1(1) and (3) with $\det(B_1) = 0$. As Ker_D and Im_D are projective R-modules of rank 1 (see [2], Lem. 3.1), the short exact sequence $0 \to \mathrm{Ker}_D \to R^2 \to \mathrm{Im}_D \to 0$ splits, i.e., it has a section $\sigma : \mathrm{Im}_D \to R^2$. Let $C_1 \in \mathbb{M}_2(R)$ be the unique matrix such that $\mathrm{Ker}_D \subseteq \mathrm{Ker}_{C_1-C}$ and $\sigma(\mathrm{Im}_D) \subseteq \mathrm{Ker}_{C_1}$. As Ker_D is a direct summand of R^2 of rank 1 and for $t \in \mathrm{Ker}_D$ we have $\mathrm{adj}(A)C_1(t) = \mathrm{adj}(A)C(t) = -t$, it follows first that $\mathrm{Ker}_D \subseteq \mathrm{Ker}_{D_1}$, second that $\mathrm{Im}_{C_1} = C_1(\mathrm{Ker}_D) = C(\mathrm{Ker}_D)$ is a direct summand of R^2 of rank 1 isomorphic to Ker_D, and third that $\mathrm{Ker}_{C_1} = \sigma(\mathrm{Im}_D)$ is also a direct summand of R^2 of rank 1. Moreover, we compute

$$\mathrm{Im}_{D_1} = D_1\big(\sigma(\mathrm{Im}_D)\big) = \{x + \mathrm{adj}(A)C_1(x) | x \in \sigma(\mathrm{Im}_D)\} = \sigma(\mathrm{Im}_D).$$

We conclude that $C_1, D_1 \in Um\big(\mathbb{M}_2(R)\big)$ and $\det(C_1) = \det(D_1) = 0$. Hence (1) $\Rightarrow$ (2), thus Theorem 1.2 holds.

4. Proof of Theorem 1.3

Part (1) holds as clearly statement (4) of [2], Thm. 4.3 implies statement (3) of Theorem 1.3. To prove part (2) we first note that A modulo $R\det(A)$ is simply extendable (see [2], Lem. 4.1(1)) and hence from [2], Prop. 5.1(1) it follows that it is non-full, i.e., there exist $\bar{l}, \bar{m}, \bar{o}, \bar{q} \in R/R\det(A)$ such that A modulo $R\det(A)$ is $\begin{bmatrix} \bar{l} \\ \bar{m} \end{bmatrix} \begin{bmatrix} \bar{o} & \bar{q} \end{bmatrix}$. If $l, m, o, q \in R$ lift $\bar{l}, \bar{m}, \bar{o}, \bar{q}$ (respectively), then $B := \begin{bmatrix} l \\ m \end{bmatrix} \begin{bmatrix} o & q \end{bmatrix}$ is congruent to A modulo $R\det(A)$ and $\det(B) = 0$, hence A is weakly determinant liftable. Thus Theorem 1.3 holds.

5. A criterion for weakly determinant liftability

Theorem 5.1. *For* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ *the following properties hold.*

(1) If A is determinant liftable, then there exists $(x, y, z, w) \in R^4$ such that $\Phi(x, y, z, w) = 0$.

(2) If there exists $(x, y, z, w) \in R^4$ such that $\Phi(x, y, z, w) = 0$, then A is weakly determinant liftable.

(3) If either $N(R) = 0$ or $\det(A) \notin Z(R)$, then the converse of part (2) holds.

(4) If $\det(A) \in Z(R)$ and A is weakly determinant liftable, then there exists $(x, y, z, w) \in R^4$ such that $\Phi(x, y, z, w) \in N(R)$.

Proof. If A is determinant liftable, then there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and $xw - yz = 0$ by Theorem 1.2, so $\Phi(x, y, z, w) = 0$. Thus part (1) holds. If $v = (x, y, z, w) \in R^4$ is as in part (2), then for $B = B_{A,v}$ we have $\det(B) = 0$ by Equation (I). Thus, as A and B are congruent modulo $R\det(A)$, A is weakly determinant liftable. The proof that part (3) holds if $N(R) = 0$ is the same as for the existence of a retraction $\mathcal{W} \to R$ in the proof of Theorem 1.2 (see the implication (1) $\Rightarrow$ (2) of Section 3). If $\det(A) \notin Z(R)$ and $B \in \mathbb{M}_2(R)$ is congruent to A modulo $R\det(A)$ with $\det(B) = 0$, then for a $v = (x, y, z, w) \in R^4$ such that $B = B_{A,v} = A\big(I_2 + \mathrm{adj}(A)C_v\big)$ (see Lemma 3.1(1)), we have $\det\big(I_2 + \mathrm{adj}(A)C_v\big) = 0$ and part (3) holds by Equation (I). Part (4) follows from part (3). $\square$

Example 5.2. If R is such that $N(R) = 0$ and there exists $A \in Um\big(\mathbb{M}_2(R)\big)$ which is not determinant liftable but is weakly determinant liftable (see Example 1.9), then there exists $(x, y, z, w) \in R^4$ such that $\Phi(x, y, z, w) = 0$ by Theorem 5.1(3). Hence the converse of Theorem 5.1(1) does not hold in general.

Remark 5.3. If $\upsilon = (x, y, z, w) \in R^4$ is such that $\Phi(x, y, z, w) \neq 0 = \Phi(x, y, z, w)^2$ and the matrix $B_{A,\upsilon}$ is not unimodular, i.e., the ideal $\mathfrak{b}$ generated by its entries is not R, then there exists $\upsilon' = (x', y', z', w') \in \upsilon + R\Phi(x, y, z, w)^4$ such that $\Phi(x', y', z', w') = 0$ iff $\Phi(x, y, z, w) \in \Phi(x, y, z, w)\mathfrak{b}$.

6. Proofs of Theorems 1.4 and 1.7

The 'if' part of Theorem 1.4 follows from the fact that all unimodular matrices in $\mathbb{M}_2(R)$ of zero determinant are determinant liftable. Based on Theorem 1.3(1), for the 'only if' part it suffices to show that if R is a Π_2 and if for $A \in Um\big(\mathbb{M}_2(R)\big)$ there exists $B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$ congruent to A modulo $R\det(A)$ and $\det(B) = 0$, then A is simply extendable. As R is a Π_2 ring, B is simply extendable. From this and [2], Thm. 4.3 it follows that there exists $(e, f) \in Um(R^2)$ such that $(a_1e + c_1f, b_1e + d_1f) \in Um(R^2)$ and so $(a_1e + c_1f, b_1e + d_1f, ad - bc) \in Um(R^2)$. As $B - A \in \mathbb{M}_2(R\det(A))$, it follows that $(ae + cf, be + df, ad - bc) \in Um(R^2)$. Thus A is simply extendable by [2], Cor. 4.7(2). So Theorem 1.4 holds.

To prove Theorem 1.7, we note that R is a Π_2 ring by [2], Thm. 1.4. Hence, based on Theorems 1.3(1) and 5.1(1) and (2), it suffices to show that if A is weakly determinant liftable, then it is extendable. Let $B \in \mathbb{M}_2(R)$ be congruent to A modulo $R\det(A)$ and $\det(B) = 0$. As B is non-full by hypothesis, A modulo $R\det(A)$ is non-full. Thus A modulo $R\det(A)$ is simply extendable (see [2], Prop. 5.1(1)) and hence A is extendable (see [2], Lem. 4.1(1)). Thus Theorem 1.7 holds.

7. On $WJ_{2,1}$ and $J_{2,1}$ rings

We first prove Theorem 1.11. Let R be a $WJ_{2,1}$ ring. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Um\big(\mathbb{M}_2(R)\big)$. By taking $(\Psi, \Delta) = (1, 0)$ in Definition 1.10(1), it follows that there exists $(x, y, z, w) \in R^4$ such that $ax + by + cz + dw = 1$ and $xw - yz = 0$, and hence A is determinant liftable by Theorem 1.2. Thus part (1) holds. We assume now that R is also a Hermite ring. The 'only if' of part (2) follows from the fact that each elementary divisor ring is an SE_2 ring (see [2], Prop. 1.3) and hence a Π_2 ring. For the 'if' of part (2), if R is also a Π_2 ring, then from part (1) and Theorem 1.4 we get that R is an SE_2 ring and hence an elementary divisor ring by [2], Cor. 1.8. Note that each Hermite domain is a Bézout domain and hence a pre-Schreier domain (see [2], Sect. 2) and a Π_2 domain (see [2], paragraph after Thm. 1.4). Hence part (2) holds. Thus Theorem 1.11 holds.

Proposition 7.1. *A ring R is a $J_{2,1}$ ring in the sense of Definition 1.10(2) iff it is a $J_{2,1}$ ring in the sense of* [8]*, Def. 4.6.*

Proof. For the 'only if' part, let $(\alpha, \beta, \gamma, \delta) \in R^4$. As R is a Hermite ring, there exist $e \in R$ and $(a, b, c, d) \in Um(R^4)$ such that $(\alpha, \beta, \gamma, \delta) = e(a, b, c, d)$. For $\Psi \in R$, the equation $\alpha X + \beta Y + \gamma Z + \delta W = \Psi$ has a solution $(x, y, z, w) \in R^4$ iff $\Psi \in Re$. Assume now $(\Psi, \Delta) \in Re \times R$. Let $f \in R$ be such that $\Psi = ef$. From Definition

1.10 applied to $(\Psi,\Delta)=(f,0)$ it follows that there exists $(x,y,z,w)\in R^4$ such that $ax+by+cz+dw=f$ and $xw-yz=\Delta$. Hence $\alpha x+\beta y+\gamma z+\delta w=ef=\Psi$ and $xw-yz=\Delta$, thus the 'only if' of part (2) holds.

For the 'if' of part (2), if R is a $J_{2,1}$ ring in the sense of [8], Def. 4.6, then clearly it is a $WJ_{2,1}$ ring and it is a Hermite ring by [8], Prop. 4.11. □

8. Rings with universal (weakly) determinant liftability

Let $GL_2(R)$ be the group of units of $\mathbb{M}_2(R)$. For a matrix $E\in\mathbb{M}_2(R)$, let $[E]\in GL_2(R)\backslash\mathbb{M}_2(R)/GL_2(R)$ be its equivalence class (double coset). For a projective R-module P of rank 1, let $[P]\in Pic(R)$ be its class.

Proposition 8.1. *We consider the following statements on R.*

(1) *For each $a\in R$, the map of sets*

$$\{B\in Um\big(\mathbb{M}_2(R)\big)|\det(B)=0\}\to\{\bar{B}\in Um(\mathbb{M}_2(R/Ra))|\det(\bar{B})=0\},$$

defined by the reduction modulo Ra, is surjective.

(2) *For each $a\in R$, the map of sets of equivalence classes*

$$\big\{[B]|B\in Um\big(\mathbb{M}_2(R)\big),\ \det(B)=0\}\to\{[\bar{B}]|\bar{B}\in Um\big(\mathbb{M}_2(R/Ra)\big),\ \det(\bar{B})=0\},$$

defined by the reduction modulo Ra, is surjective.

(3) *For each $a\in R$, every projective R/Ra-module of rank 1 generated by 2 elements is isomorphic to the reduction modulo Ra of a projective R-module of rank 1 generated by 2 elements.*

(4) *Each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable.*

Then $(1)\Rightarrow(2)\Leftrightarrow(3)$ *and* $(1)\Rightarrow(4)$. *If $sr(R)\le 4$, then* $(1)\Leftrightarrow(4)$.

Proof. For a pair $\pi:=(P,Q)$ of projective R-submodules of R^2 of rank 1 and generated by 2 elements such that we have a direct sum decomposition $R^2=P\oplus Q$, let $E_\pi\in Um\big(\mathbb{M}_2(R)\big)$ be the projection on P along Q; so $\det(E_\pi)=0$, P and Q are dual to each other (i.e., $[Q]=-[P]$, with $Pic(R)$ viewed additively), and $U_{[P]}:=[E_\pi]$ depends only on $[P]$. Each projective R-module of rank 1 generated by 2 elements is isomorphic to such a P. For $F\in Um\big(\mathbb{M}_2(R)\big)$ with $\det(F)=0$, Ker_F and Im_F are projective R-module of rank 1 generated by 2 elements and the short exact $0\to\mathrm{Ker}_F\to R^2\to\mathrm{Im}_F\to 0$ has a section $\varphi:\mathrm{Im}_F\to R^2$ (see [2], Lem. 2.1); if $\tau_F:=\big(\mathrm{Im}(\varphi),\mathrm{Ker}_F\big)$, then $[F]=[E_{\tau_F}]=U_{[\mathrm{Im}_F]}$. Thus

$$\big\{[B]|B\in Um\big(\mathbb{M}_2(R)\big),\ \det(B)=0\}=\{U_{[P]}|P\oplus Q=R^2,\ P\text{ has rank }1\}.$$

From this and its analogue over R/Ra, it follows that $(2)\Leftrightarrow(3)$. Clearly, $(1)\Rightarrow(2)$.

For $(1)\Rightarrow(4)$, let $A\in Um\big(\mathbb{M}_2(R)\big)$. By applying (1) to $a=\det(A)$ and the reduction $\bar{B}$ of A modulo Ra, it follows that there exists $B\in Um\big(\mathbb{M}_2(R)\big)$ congruent to A modulo $R\det(A)$ and $\det(B)=0$, so A is determinant liftable.

Assume $sr(R)\le 4$. To prove $(4)\Rightarrow(1)$, let $a\in R$. Let $\bar{B}\in Um\big(\mathbb{M}_2(R/Ra)\big)$ with $\det(\bar{B})=0$. Let $C\in Um\big(\mathbb{M}_2(R)\big)$ be such that its reduction modulo Ra is $\bar{B}$ by [2], Prop. 2.4(1); we have $\det(C)\in Ra$. As C is determinant liftable, there exists $B\in Um\big(\mathbb{M}_2(R)\big)$ with $\det(B)=0$ and congruent to C modulo $R\det(C)$ and hence also modulo Ra; so the map of statement (1) is surjective, hence $(4)\Rightarrow(1)$. □

Example 8.2. If R is an integral domain of dimension 1, then each matrix $A \in Um\big(\mathbb{M}_2(R)\big)$ is determinant liftable. To check this we can assume that $\det(A) \neq 0$ and this case follows from [2], Thm. 1.7(1) and Theorem 1.3(1).

Proposition 8.3. *We consider the following two statements on R.*

(1) *For each $a \in R$, $Um\big(\mathbb{M}_2(R/Ra)\big)$ is contained in the image of the modulo Ra reduction map $\{B \in \mathbb{M}_2(R) | \det(B) = 0\} \to \{\bar{B} \in \mathbb{M}_2(R/Ra) | \det(\bar{B}) = 0\}$.*

(2) *Each matrix in $Um\big(\mathbb{M}_2(R)\big)$ is weakly determinant liftable.*

Then (1) $\Rightarrow$ (2), and the converse holds if $sr(R) \leq 4$.

Proof. It is the same as the last two paragraphs of the proof of Proposition 8.1, with determinant and $B \in Um\big(\mathbb{M}_2(R)\big)$ replaced by weakly determinant and $B \in \mathbb{M}_2(R)$ (respectively). □

9. A criterion for determinant liftability via completions

The following proposition is likely to be well-known.

Proposition 9.1. *Let $A \in Um\big(\mathbb{M}_2(R)\big)$, let $t \in R$ be such that $\det(A) \in Rt$ and let $\hat{R}$ be the t-adic completion of R. Then there exists $B \in Um\big(\mathbb{M}_2(\hat{R})\big)$ whose reduction modulo $\mathrm{Ker}(\hat{R} \to R/Rt)$ is the reduction of A modulo Rt and $\det(B) = 0$.*

Proof. Let $B_0 := A$. By induction on $n \in \mathbb{N}$, we show that there exists $B_n \in \mathbb{M}_2(R)$ congruent to B_{n-1} modulo $Rt^{2^{n-1}}$ and $\det(B_n) \in Rt^{2^n}$. For $n = 1$, let $s \in R$ be such that $\det(A) = st$. With $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, for $B_1 := A + t \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \mathbb{M}_2(R)$, $\det(B_1)$ is congruent modulo Rt^2 to $st + (dx + cy + bz + aw)t$. As $A \in Um\big(\mathbb{M}_2(R)\big)$, the linear equation $dx + cy + bz + aw = -s$ has a solution $(x, y, z, w) \in R^4$; for such a solution we have $\det(B_1) \in Rt^2$. The passage from n to $n+1$ follows from the case $n = 1$ applied to $(B_n, Rt^{2^{n+1}})$ instead of (A, Rt^2). This completes the induction. As t belongs to the Jacobson radical $J(\hat{R})$ of $\hat{R}$, the limit $B \in \mathbb{M}_2(\hat{R})$ of the sequence $(B_n)_{n\geq 1}$ exists. Clearly, $\det(B) = 0$. As $\mathrm{Ker}(\hat{R} \to R/Rt) \subset J(\hat{R})$, we have $B \in Um\big(\mathbb{M}_2(\hat{R})\big)$. □

Proposition 8.3 also follows from the smoothness part of Theorem 2.1(2) via a standard lifting argument. Proposition 8.3 gives directly the following result.

Corollary 9.2. *Let $A \in Um\big(\mathbb{M}_2(R)\big)$. If R is complete in the $\det(A)$-adic topology, then A is determinant liftable.*

Acknowledgement. The third author would like to thank SUNY Binghamton for good working conditions and Ofer Gabber for sharing the argument of the footnote.

References

[1] H. Bass, E. H. Connell, D. Wright *Locally polynomial algebras are symmetric algebras.* Invent. Math. **38** (1977), no. 3, 279–299.

[2] G. Călugăreanu, H. F. Pop, A. Vasiu *Matrix invertible extensions over commutative rings. Part I: general theory*, 18 pages final version to appear in J. Pure Appl. Algebra, `https://arxiv.org/abs/2404.05780`.

[3] J. Fresnel, *Points entiers de certains schémas de matrices. [Integer points of certain matrix schemes],* Arch. Math. (Basel) **100** (2013), no. **6**, 521–531.

[4] R. Gilmer, R. C. Heitmann *On Pic(R[X]) for R seminormal.* J. Pure Appl. Algebra **16** (1980), no. 3, 251–257.
[5] C. U. Jensen *Arithmetical rings.* Acta Math. Acad. Sei. Hungar. **17** (1966), 115–123.
[6] I. Kaplansky *Elementary divisors and modules.* Trans. Amer. Math. Soc. **66** (1949), 464–491.
[7] T. Y. Lam *Serre's problem on projective modules.* Springer Monogr. Math., Springer-Verlag, Berlin, 2006.
[8] D. Lorenzini *Elementary divisor domains and Bézout domains.* J. Algebra **371** (2012), 609–619.
[9] D. Quillen *Projective modules over polynomial rings.* Invent. Math. **36** (1976), 167–171.
[10] *Revêtements Étales et groupe fondamental. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1).* Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Math., Vol. **224**, Springer-Verlag, Berlin-New York, 1971.
[11] R. Wiegand, S. Wiegand *Finitely generated modules over Bézout rings.* Pacific J. Math. **58** (1975), no. 2, 655–664.

Grigore Călugăreanu E-mail: calu@math.ubbcluj.ro
Address: Department of Mathematics, Babeş-Bolyai University,
1 Mihail Kogălniceanu Street, Cluj-Napoca 400084, Romania.

Horia F. Pop E-mail: horia.pop@ubbcluj.ro
Address: Department of Computer Science, Babeş-Bolyai University,
1 Mihail Kogălniceanu Street, Cluj-Napoca 400084, Romania.

Adrian Vasiu, E-mail: avasiu@binghamton.edu
Address: Department of Mathematics and Statistics, Binghamton University,
P. O. Box 6000, Binghamton, New York 13902-6000, U.S.A.