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# Abelian groups with C2

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Almost all Abelian groups with the property that each subgroup isomorphic to a direct summand, is also a direct summand, are determined. The relationship with co-Hopfian groups is also addressed.

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## 1. Introduction

This note is about (additively written) Abelian groups, that is, in the sequel the word "group" always means "Abelian group".

For any group $G$ we denote by $T(G)$ the torsion subgroup of $G$ and by $D(G)$ the maximum divisible subgroup of $G$. For any prime number $p$, $T_p$ denotes the $p$-component of $T(G)$, and by the *torsion spectrum* of $G$ we mean the set of primes $p$ such that $T_p \neq 0$.

A group is called *homococyclic* if it is the direct sum of copies of a single *cocyclic* group (i.e. either $\mathbb{Z}_p^\infty$ or $\mathbb{Z}_p^k$ for some positive integer $k$).

For any group $G$, as usually $X \subseteq G$ shows $X$ is a *subset* of $G$ but $X \leq G$ is used only for a *subgroup* $X$ of $G$. $\mathbb{P}$ denotes the set of all prime numbers. The word "summand" will be used only for "direct summands". The notation used for this is $\leq^\oplus$.

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For unexplained notions and results, we refer the reader to Laszlo Fuchs's treatise on Infinite Abelian groups ([1, 2]) or the newest one ([3]). Whenever it is more convenient, we will use the widely accepted shorthand "iff" for "if and only if" in the text.

In his work on continuous rings (see [8]), half a century ago, Utumi identified three conditions on a ring that are satisfied if the ring is self-injective. These conditions were extended to modules by Mohamed and Müller (see [5]), starting with the observation (Proposition 2.1) that any (quasi)-injective module satisfies all these conditions.

**Definition.** A module $M$ is called a *C1-module*, if every submodule is essential in a direct summand of $M$.

$M$ is called a *C2-module*, if whenever $A$ and $B$ are submodules of $M$ with $A \cong B$ and $B \leq^{\oplus} M$ then $A \leq^{\oplus} M$.

$M$ is called a *C3-module*, if whenever $A$ and $B$ are submodules of $M$ with $A \leq^{\oplus} M$, $B \leq^{\oplus} M$ and $A \cap B = 0$, then $A \oplus B \leq^{\oplus} M$.

We shall also use the following.

**Definitions.** For a prime $p$, a group $H$ is called *$p$-automorphic* if multiplication by $p$ is an automorphism. Similarly, if $T$ is a torsion group, we say that a group $H$ is *$T$-automorphic* if it is $p$-automorphic for all primes $p$ in the torsion spectrum of $T$.

The goal of this note is to characterize, *as much as possible*, the classes of C2 Abelian groups. In doing this, the relationship between C2 Abelian groups and co-Hopfian Abelian groups is also addressed.

An explanation is due with respect to our "as much as possible" above.

It is well-known for Abelian groups theorists that the characterization of endoregular groups was possible (e.g. see [2, Proposition 112.7]) only for not reduced groups and for torsion groups. For reduced groups, the torsion part is elementary, the factor group $G/T(G)$ is divisible and the group is between the direct sum and the direct product of their $p$-components. We quote from [2]: "A satisfactory, more or less explicit description of reduced groups with regular endomorphism rings seems to be a hard problem. Manifestly, the difficulty lies in singling out the suitable mixed groups between the direct sum and the direct product of their $p$-components".

It is readily seen that groups with (von Neumann) regular endomorphism ring (called endoregular in the sequel) are C2. Therefore, one expects to have only partial results for reduced groups which have C2. Essentially, if $T$ is a direct sum of $p$-homococyclic groups and $P$ is the corresponding direct product, there are going to be a huge number of groups $G$ between $T$ and $P$ such that $G/T$ is divisible. Getting a handle on which ones are C2, or even co-Hopfian, seems like a very difficult task.

Our main results are

(A) A torsion-free group is C2 iff it is divisible.
(B) A torsion group is C2 iff it has homococyclic or divisible $p$-components.

(C) If $p$ is a prime, then $G$ is a C2 group iff $G = T_p \oplus G_p$, where $T_p$ is homococyclic, and $G_p$ is a $p$-automorphic C2 group.
(D) If $E$ is the maximum divisible subgroup of $T(G)$ then $G$ is C2 iff $G$ is a direct sum $E \oplus H$, where $H$ is an $E$-automorphic C2 group.
(E) If $G$ is torsion-reduced and $G = H \oplus D(G)$, then $G$ is C2 iff $H$ is C2.
(F) Suppose $G$ is a reduced group such that $T_p$ is finite for all $p$ and $G/T(G)$ is divisible.
   (a) If $G/T(G)$ has finite rank, then $G$ is co-Hopfian.
   (b) If $G$ is C2, then $G$ is co-Hopfian.
(G) If $G$ is reduced, $G/T(G)$ is divisible and $T_p$ is cyclic for all primes $p$ then $G$ is C2 iff it is co-Hopfian.
(H) If the reduced group $G$ has cyclic $p$-torsion for all prime $p$ and $G/T(G)$ is divisible of finite rank then $G$ is C2. Examples show that both hypothesis are necessary.

## 2. Known Facts About C2 Modules

The notion of direct-injective modules was introduced by Nicholson [6], as a generalization of quasi-injective modules.

A right $R$-module $M$ is *direct-injective* if given a direct summand $N$ of $M$ with inclusion $i_N : N \to M$ and any monomorphism $g : N \to M$ there exist $f \in \mathrm{End}_R(M)$ such that $f \circ g = i_N$.

Nicholson and Yousif ([7, Theorem 7.13]) showed that the *class of direct-injective modules is equivalent to the class of C2-modules*. Specifically, a module $M$ is C2 if and only if monomorphisms in $\mathrm{End}_R(M)$ are isomorphisms. Using the modularity of the submodule lattice of any module it was readily proved that *direct summands of C2 modules are C2*.

Some properties of direct-injective modules are gathered in [4].

A module $M$ over a ring $R$ is said to be *divisible* if $rM = M$ for all not right zero-divisor elements $r \in R$. The following result was proved in [4].

**Proposition 1 ([2.9]).** *Let $R$ be a commutative domain and $M$ be a torsion-free module. Then $M$ is a direct-injective module iff $M$ is a divisible module.*

From the same reference we also mention the following.

**Corollary 2 ([4, 3.4]).** *If $M$ is a finitely generated direct-injective module over a principal ideal domain, then the torsion submodule of $M$ is a direct-injective module.*

An important class of C2 modules are the *endoregular* modules (i.e. modules $M$ such that $\mathrm{End}_R(M)$ is a von Neumann regular ring). It is well-known that *every endoregular module is a direct-injective (i.e. C2) module* but the converse need not be true.

## 3. C2 Abelian Groups

First recall that the (Abelian) group $G$ is C2 if whenever $M$ is a summand of $G$ and $N$ is a subgroup of $G$ isomorphic to $M$, then $N$ is also a summand of $G$.

Note that every divisible group is injective (so quasi-injective) so C2. For torsion-free groups, the converse also holds. Indeed, from Proposition 1 (previous section) we derive the following.

**Corollary 3.** *Let $G$ be a torsion-free group. The following conditions are equivalent*:

(i) *$G$ is C2*;
(ii) *$G$ is direct-injective*;
(iii) *$G$ is injective* (*divisible*);
(iv) *$G$ is divisible.*

Since $\mathbb{Z}$ is not C2, by denial we infer that any group with a direct summand isomorphic to $\mathbb{Z}$ is not C2.

**Corollary 4.** *The only indecomposable C2 groups are the cocyclic groups and $\mathbb{Q}$.*

The other module theoretic results mentioned in the previous section yield some information on our subject.

(1) The torsion subgroup $T(G)$ of a finitely generated C2 group $G$ is also C2.
(2) Since much information is known about Abelian groups whose endomorphism ring is regular (for a group $G$, $\mathrm{End}(G)$ is regular iff images and kernels of endomorphisms are direct summands), the following theorem (see [2]) gives examples of classes of C2 groups.

**Theorem 5.** (a) *If $G$ is a torsion group then* $\mathrm{End}(G)$ *is regular iff $G$ is an elementary group.*

(b) *If $G$ is a non-reduced group then* $\mathrm{End}(G)$ *is regular iff it is a direct sum of a divisible torsion-free group and an elementary group.*

Due to the above first corollary, *only torsion and mixed C2 groups must be determined*, of course, now by Abelian group theory specific methods. The only result we use from the more general case of modules is that *summands of C2 groups are also C2*.

It is just routine to show that *a torsion group $G$ is C2 iff every $p$-component $G_p$ is C2*, and that *a $p$ -group is C2 iff it is homococyclic or divisible*. Therefore we have the following theorem.

**Theorem 6.** *A torsion group is C2 iff it has homococyclic or divisible primary components.*

Since, in the sequel, this follows from our (more general) considerations, we do not provide a proof here.

**Remark.** As already mentioned in the introduction, (quasi-)injective modules are C2. In the Abelian group setting, recall that the quasi-injective Abelian groups are known: these are divisible *or* torsion with homococyclic components.

**Examples.** (1) $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not C2, so *direct sums of C2 groups may not be C2*. The corresponding example in [7] is: If $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$, $A = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}$ where $F$ is a field, then $R_R$ is not a C2 module but $R = A \oplus B$ and both $A_R$ and $B_R$ are C2 modules.

(2) $\mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_p$ is not C2, but both summands are.
(3) $\mathbb{Z}_{2^\infty} \oplus \mathbb{Z}_3$ is C2, but not quasi-injective.

We start with the below useful discussion.

**Lemma 7.** *Suppose $p$ is a prime, $G$ is a group and there is a splitting $G = T_p \oplus G_p$. If $M$ is a summand of $G$ with torsion $U_p$, then there is a decomposition $M = U_p \oplus M_p$ which extends to a decomposition $G = W_p \oplus U_p \oplus M_p \oplus L_p$, where $T_p = W_p \oplus U_p$, and in addition, $G_p = M_p \oplus L_p$.*

**Proof.** Suppose $G = M \oplus L$. Since $U_p$ is a summand of $T_p$, which is a summand of $G$, $U_p$ will also be a summand of $M$. So there is a decomposition $M = U_p \oplus M_p$. Similarly, if $W_p$ is the $p$-torsion-subgroup of $L$, then there is a decomposition $L = W_p \oplus L_p$. The conclusions now easily follow. □

**Theorem 8.** *If $p$ is a prime, then $G$ is a C2 group iff $G = T_p \oplus G_p$, where $T_p$ is homococyclic, and $G_p$ is a $p$-automorphic C2 group.*

**Proof.** Fix the prime $p$. Assume first that $G$ is a C2-group. If $T_p$ is not of the stated form, it follows that there are distinct values in the set $\{1, 2, 3, \ldots, \infty\}$, say $m < n$, such that $T_p$ has a summand of the form $C_m := \mathbb{Z}_{p^m}$ and a summand of the form $C_n := \mathbb{Z}_{p^n}$. Note that $C_m$ is pure in $T$, and hence pure in $G$ and clearly $C_m$ is bounded (by $p^m$). It follows that $C_m$ is a summand of $G$. On the other hand, there is a proper subgroup $C'_m < C_n$ isomorphic to $C_m$. Since $C'_m$ is not a summand of $C_n$, it is certainly not a summand of $G$. Since $T_p$ is either divisible or bounded, it must be a summand of $G$, say $G = T_p \oplus G_p$. As $G_p$ is a summand of $G$, it must be C2. In addition, $\phi : G_p \to G_p$ given by $\phi(x) = px$ is clearly injective. Therefore, $pG_p \cong G_p$ is isomorphic to a summand of $G$. So, $pG_p$ is a summand of $G_p$. However, since $G_p$ has no $p$-torsion and $p(G_p/pG_p) = 0$, it follows that $G_p = pG_p$, as required.

Conversely, suppose there is a splitting, as indicated. Let $M$ be some summand of $G$ and $N$ be a subgroup isomorphic to $M$; we need to show $N$ is also a summand of $G$. Consider a decomposition $G = W_p \oplus U_p \oplus M_p \oplus L_p$, as in Lemma 7. In particular, $W_p$ and $U_p$ are homococyclic and $M_p$ and $L_p$ are $p$-automorphic.

Let $V_p \oplus N_p$ be the corresponding decomposition of $N$, so $V_p \cong U_p$ and $N_p \cong M_p$.

**Case 1.** $T_p$ is bounded (and homococyclic). In particular, if $\pi : G \to T_p$ is the projection with kernel $G_p$, then since $M_p$ and $N_p$ are $p$-automorphic and $T_p$ is bounded, we have $\pi(M_p) = \pi(N_p) = 0$. Therefore, $M_p$ and $N_p$ are actually subgroups of $G_p$. Since $M_p$ is a summand of $G$, it is also a summand of $G_p$. Therefore, $N_p$ is also a summand of $G_p$. Since $V_p$ is trivially a summand of $T_p$, it follows that $N$ is a summand of $G$, as required.

**Case 2.** $T_p$ is divisible. Since $N_p \cap T_p = 0$, it follows that there is a splitting $G = T_p \oplus G'_p$, such that $N_p \leq G'_p$. Replacing $G_p$ by $G'_p$, there is no loss of generality in assuming that $N_p \leq G_p$. By Lemma 7, $M_p$ is isomorphic to a summand of $G_p$. Therefore, since $G_p$ is assumed to be C2, $N_p$ is a summand of $G_p$. And since $V_p$ is clearly a summand of $T_p$, it follows that $N$ is a summand of $G$, as required. □

Though the next result follows from the preceding one, it is actually pretty easy to verify on its own.

**Corollary 9.** *If $G$ is a $C2$ group, then for any $p$ such that $T_p = 0$, $G$ is $p$-automorphic. In other words, whenever multiplication by $p$ is injective, it is bijective.*

**Corollary 10.** *If $T_p = 0$ for all but finitely many primes, then $G$ is $C2$ iff it is isomorphic to a direct sum*

$$T_{p_1} \oplus \cdots \oplus T_{p_k} \oplus D,$$

*where each $T_{p_j}$ is homococyclic and $D$ is torsion-free divisible.*

We turn to another reductive result.

**Lemma 11.** *If $E$ is the maximum divisible subgroup of $T(G)$ then $G$ is $C2$ iff $G$ is a direct sum $E \oplus H$, where $H$ is an $E$-automorphic $C2$ group.*

**Proof.** We certainly have such a decomposition $G = E \oplus H$. Clearly, if $G$ is C2, then so is $H$. And if $T_p$ is divisible, then there is a decomposition $G = E \oplus H = T_p \oplus E_p \oplus H$. Since $E_p \oplus H$ is a C2 group without $p$-torsion, it must be $p$-automorphic, which clearly implies that $H$ is also $p$-automorphic.

Conversely, suppose $H$ is an $E$-automorphic C2 group. Again, suppose $M$ is a summand of $G$, and $N \leq G$ is isomorphic to $M$. If $E_M = E \cap M$, then $E_M$ must be divisible, so there is a decomposition $M = E_M \oplus H_M$. By Lemma 7, $H_M$ is isomorphic to a summand of $H$.

Let $N = E_N \oplus H_N$ be the corresponding decomposition of $N$. It therefore follows that $H_N \cap E = 0$, so we may choose the summand $H$ so that it contains $H_N$. Since $H$ is C2, it follows that $H_N$ is a summand of $H$, which easily implies that $N$ is a summand of $G$, as required. □

So, in our study of C2 groups, we may as well restrict our attention to those whose torsion subgroup is reduced, which we abbreviate to *torsion-reduced*.

**Lemma 12.** *Suppose $G$ is any torsion-reduced group and each $T_p$ is bounded. Then the Ulm subgroup $G^1$ agrees with the maximum divisible subgroup of $G$.*

**Proof.** Certainly, every element of $D(G)$ must be in $G^1$, the question is the reverse containment. So, assume $a \in G^1$ is nonzero; we want to show $a \in D(G)$.

For each prime $p$, choose a non-negative integer $k_p$ such that $p^{k_p} T_p = 0$. For every $n = 0, 1, 2, \ldots$, choose $c_n$ such that $p^{k_p+n} c_n = x$. Let $b_{p,n} = p^{k_p} c_n$. Clearly, $b_{p,0} = x$. For $n \geq 0$, since $p^{k_p+n}(c_n - pc_{n+1}) = x - x = 0$, it follows that

$$b_{p,n} - pb_{p,n+1} = p^{k_p}(c_n - pc_{n+1}) = 0.$$

Therefore $Q$, the subgroup of $G$ generated by all of the $b_{p,n}$ where $p$ ranges over all primes and $n$ ranges over all non-negative integers, is isomorphic to $\mathbb{Q}$. So $x \in D(G)$, as required. □

**Proposition 13.** *If $G$ is torsion-reduced and $G = H \oplus D(G)$, then $G$ is $C2$ iff $H$ is $C2$.*

**Proof.** Certainly if $G$ is C2, then so is $H$. So assume that $H$ is C2. Let $M$ be a summand of $G$ and $N$ be a subgroup of $G$ isomorphic to $M$; we need to show $N$ is also a summand.

Note that $H$ modulo its torsion is torsion-free divisible. Therefore, the same holds for $G$, and hence its summand $M$, and in addition, its isomorphic copy $N$. We next claim that $N$ must be a pure subgroup of $G$. Since the $p$-components of the torsion subgroup of $G$, which is also the torsion subgroup of $H$, are homococyclic, it follows that $T(N)$, the torsion subgroup of $N$ is, in fact, a summand of $T(G)$. Therefore, $T(N)$ is also pure in $G$. In addition, since $N/T(N)$ is divisible, it is pure in $G/T(N)$ (in fact, it is a summand). Therefore, $N$ is pure in $G$, as claimed.

It follows that $N^1$, the divisible subgroup of $N$, agrees with

$$N \cap G^1 = N \cap D(G).$$

Therefore, there is a splitting $N = H_N \oplus N^1$, and $H_N \cap D = 0$. As a result, there is no loss of generality in assuming that $H_N \leq H$. So, as above, since $H_M$ is isomorphic to a summand of $H$ and $H_N$ is isomorphic to $H_M$, it follows that $H_N$ is a summand of $H$. Therefore, $N$ is a summand of $G$, as required. □

So, by the above, to decide if $G$ is C2, there is no loss of generality in assuming that $G$ is reduced, the $p$-components of its torsion subgroup $T(G)$ are homococyclic and $G/T(G)$ is divisible (and torsion-free). In particular, the Ulm subgroup $G^1 = 0$.

The above discussion implies that $G$ embeds as a pure subgroup of its $p$-adic completion $\widehat{G}$, and

$$\widehat{G} = \prod_{p \in \mathbb{P}} T_p = \widehat{T(G)}.$$

We mention the *two extreme cases*: If $G = T(G)$ is torsion with homococyclic $p$-components, then $G$ is trivially C2.

On the other hand, suppose $G = \widehat{T}$ is algebraically compact. If $M$ is any summand of $G$, then $M$, too, is algebraically compact, and so, too, is any subgroup $N$, isomorphic to $M$. But we saw in the above discussion that any subgroup of $G$ that is divisible modulo its torsion is necessarily pure in $G$. Since algebraically compact groups are pure-injectives, it follows that $N$ is a summand of $G$, as required.

In closing, we have the following example.

**Example 14.** Pure subgroups of C2 groups may not be C2.

Indeed, suppose $B$ is the direct sum of copies of $\mathbb{Z}_p$ for all primes and $C$ is its $p$-adic completion. Arguing as above, it readily follows that $G = C \oplus C$ is C2. Note that $G$ contains $H = B \oplus C$ as a pure subgroup, and Proposition 13 shows that $H$ fails to be C2.

The following remains open.

**Open question.** C2 is closed to direct summands and not closed to pure subgroups. What about closure to fully invariant subgroups?

**Remark.** If $p$ is a prime and we consider the corresponding $p$-local problem, i.e. modules over the ring of integers localized at $p$, then the $p$-local group $G$ is C2 iff $G$ is isomorphic to $T \oplus D$, where $T$ is homococyclic and $D$ is torsion-free divisible. A similar statement holds for modules over the integers localized at any finite set of primes. It is only in the case where there are infinite number of primes that $T$ will not equal $P$ (the corresponding direct product), so that there will be lots of intermediate groups $G$.

## 4. C2 Versus Co-Hopfian

We turn to a few observations regarding the relationship between the condition of being C2 and the condition of being co-Hopfian, especially in the case where $T_p$ is finite for all primes $p$.

Recall that a module $M$ is called *co-Hopfian* if every monomorphism from $M$ into $M$ is an automorphism.

**Proposition 15.** *Suppose $G$ is a reduced group such that $T_p$ is finite for all $p$ and $G/T(G)$ is divisible.*

(a) *If $G/T(G)$ has finite rank, then $G$ is co-Hopfian.*
(b) *If $G$ is C2, then $G$ is co-Hopfian.*

**Proof.** Let $\phi : G \to G$ be an injective endomorphism. First of all, it clearly is an automorphism when restricted to $T(G)$.

In part (a), $\phi$ determines an injective function $\widehat{\phi} : G/T(G) \to G/T(G)$. Since this group is a finite dimensional rational vector space, $\widehat{\phi}$ is also an automorphism, which implies that $\phi$ is an automorphism.

In part (b), $N := \phi(G) \cong G$ must be a summand of $G$ containing $T(G)$. But since $G/T(G)$ is divisible and $G$ is reduced, we can conclude that $N = G$, completing the argument. □

The converse of Proposition 15(a) is clearly not true. Suppose $T_p$ is finite for all primes $p$ and $G$ is the $p$-adic completion of $G$. It follows that $G$ is C2, and by (b), co-Hopfian. On the other hand, $G/T(G)$ clearly does not have finite rank.

Considering the converse to Proposition 15(b), there is one circumstance in which it does hold.

**Proposition 16.** *Suppose $G$ is reduced, $G/T(G)$ is divisible and $T_p$ is cyclic for all primes $p$. Then $G$ is C2 iff it is co-Hopfian.*

**Proof.** One direction being simply Proposition 15(b), suppose $G$ is co-Hopfian; we want to show that it is C2. As usual, let $M$ be a summand of $G$ and $N \leq G$ be isomorphic to $M$. Since the torsion subgroups of $M$ and $N$ are isomorphic, our condition on $T(G)$ implies that they must be, in fact, equal; denote this by $S \leq T(G)$. Now, since $M/S$ is the maximum divisible subgroup of $G/S$ and $N/S$ is divisible, it follows that we must have that $N \leq M$. Suppose $G = M \oplus K$. The function $\phi : G \to G$ which is the isomorphism $M \cong N$ on the first summand and the identity on the second is clearly injective. It follows from our hypothesis that it is also surjective. Therefore, $M = N$ is a summand of $G$, as required. □

**Corollary 17.** *Suppose the reduced group $G$ has cyclic $p$-torsion for all prime $p$, and $G/T(G)$ is divisible of finite rank. Then $G$ is C2.*

We next observe that the condition in Corollary 17, that $T_p$ is cyclic, is necessary.

**Example 18.** Suppose $T_p = \mathbb{Z}_p$ for all primes $p$ and $H$ is a group such that $T \leq H \leq \widehat{T}$ and $H/T = \mathbb{Q}$. Then $G = H \oplus T$ is co-Hopfian, but it is not C2.

**Proof.** By Proposition 15, $G$ is co-Hopfian. On the other hand, $M := 0 \oplus T$ is clearly a summand of $G$ and there is an isomorphism $M \cong T \cong N := T \oplus 0$. However, if $N$ is a summand of $G$, then it is also a summand of $H$, which clearly cannot happen since $H$ is reduced and $H/T$ is divisible. Therefore, $G$ is not C2, as stated. □

In fact, the last example can be generalized as follows.

**Proposition 19.** *Suppose $G$ is a reduced C2 group. Then $G$ is torsion iff it has a torsion summand $H$ with the same torsion spectrum as $G$.*

**Proof.** Clearly, if $G$ is torsion, we can just let $H = G$. So suppose we are given the torsion summand $H$ as stated. Passing to a summand of $H$, there is no loss of generality in assuming all $p$-components of $H$ are, in fact, cyclic. Suppose $G = H \oplus K$; we need to show that $K$ is also torsion. Let $T(K) = \bigoplus_p T_p$ be the torsion subgroup of $K$. If $x \in K \setminus T(K)$, then let $x = (y_p)$ such that $y_p \in T_p$. Let $\mathcal{P}$ be the collection of primes such that $y_p \neq 0$. There is a summand $M$ of $H$ with $\mathcal{P}$ as its torsion spectrum. If $N = \langle y_p \rangle$, there is clearly an isomorphism $M \to N$. However,

$N$ cannot be a summand of $G$ since it is not closed in $G$ in the $p$-adic topology of $G$. This contradiction completes the argument. □

We now show that in Corollary 17, the condition that $G/T$ has finite rank, is also necessary.

**Example 20.** There is a reduced group $G$ such that $T_p$ is cyclic for all primes $p$ and $G/T(G)$ is divisible that is not co-Hopfian. In particular, it is not C2.

**Proof.** For $n = 1, 2, 3, \dots$, choose a strictly increasing sequence of primes $p_n$ such that $n^{n+2} < p_n$. We let $T = \bigoplus_{0<n} \mathbb{Z}_{p_n}$, and $P = \prod_{0<n} \mathbb{Z}_{p_n}$.

Let $N \in P$ be the vector $(n)_{0<n} \in P$, where we are identifying the integer $n$ with the corresponding element of $\mathbb{Z}_{p_n}$. So $N^0$ is the constant vector $(1)_{0<n}$ and for $k = 1, 2, 3, \dots$, $N^k$ is the vector $(n^k)_{0<n} \in P$. We claim that $(1)_{0<n}, N^1, N^2, N^3, \dots$, represent linearly independent elements of the divisible group $P/T$. If this failed, we could find a positive integer $k$ and integers $a_0, \dots, a_k$ (not all equaling 0) such that $a_0(1)_{0<n} + a_1N^1 + a_2N^2 + \cdots + a_kN^k = (a_0 + a_1n + a_2n^2 + \cdots + a_kn^k)_{0<n} \in T$.

In other words, the vector $(a_0 + a_1n + a_2n^2 + \cdots + a_kn^k)_{0<n}$ has at most a finite number of nonzero-coordinates.

Suppose this expression is 0 for all coordinates exceeding the $j$th. This implies that for all $n > j$, $p_n$ divides $a_0 + a_1n + a_2n^2 + \cdots + a_kn^k$. If we consider all $n$ such that

$$n > \max\{j, k+1, |a_0|, |a_1|, \dots, |a_k|\},$$

then we will have

$$|a_0 + a_1n + a_2n^2 + \cdots + a_kn^k| < n \cdot \ n \cdot n^n = n^{n+2} < p_n.$$

It follows that for all but finitely many $n$, the integer equation

$$a_0 + a_1n + a_2n^2 + \cdots + a_kn^k = 0,$$

must hold. But the polynomial $a_0 + a_1X + a_2X^2 + \cdots + a_kX^k$ can have only a finite number of distinct roots.

Define $G$ to be the purification in $P$ of

$$T + \langle N^k : k = 0, 1, 2, \dots, i\rangle.$$

We next define $\phi : P \to P$ to be multiplication by $N$, i.e. if $(x_n)_{0<n}$ is some vector in $P$, then $\phi((x_n)_{0<n}) = (nx_n)_{0<n}$. Note that since for all $n > 0$ we have $p_n > n$, we can conclude that $\phi$ is an automorphism of $P$. In addition, for each $k = 0, 1, 2, \dots$, we have $\phi(N^k) = N^{k+1}$. This means that $\phi$ restricts to an injective, but not surjective endomorphism of $G$ (since $(1)_{0<n}$ is not in the image), so that $G$ is not co-Hopfian. □

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