"BEING A UNIT" IS AN ABAB-COMPATIBLE PROPERTY

In Diesl's dissertation thesis (2006) the following notion is considered.

Definition 2.7. Let $\mathcal{P}$ be a property that a ring element can satisfy. We will call $\mathcal{P}$ an *ABAB-compatible* property if it satisfies the following three conditions.

(1) If $a \in R$ has property $\mathcal{P}$, then $-a$ has property $\mathcal{P}$.

(2) If $a \in R$ has property $\mathcal{P}$ and e is any idempotent in R such that $ea = ae$, then $eae \in eRe$ has property $\mathcal{P}$.

(3) If a is in R and $e \in R$ is an idempotent such that $ea = ae$, and if the elements $eae \in eRe$ and $(1-e)a(1-e) \in (1-e)R(1-e)$ both have property $\mathcal{P}$ as elements of the respective corner rings, then a has property $\mathcal{P}$ in R.

Diesl's choice of the term "ABAB-compatible" is motivated by the following lemma.

Lemma 2.8. Let R be a ring, and let M_R be a right R-module. Let $\mathcal{P}$ be an ABAB-compatible property. An endomorphism $\phi \in End(M_R)$ is then the sum of an idempotent e and an element a with property $\mathcal{P}$ such that $ae = ea$ if and only if there exists a direct sum decomposition $M = A \oplus B$ such that $\phi|A$ is an element of $End(A)$ with property $\mathcal{P}$ and $(1-\phi)|B$ is an element of $End(B)$ with property $\mathcal{P}$.

As customary, for any ring R, $U(R)$ denotes the set of all the units of R. For an idempotent e, we denote by $\overline{e} = 1 - e$, the complementary idempotent. For any element a, $ae = ea$ is equivalent to $a\overline{e} = \overline{e}a$.

No details where given in presenting the following example.

Lemma 1. *(i) "Being a unit" is ABAB-compatible.*

Proof. (i) (1) obvious.

(2) If $a \in U(R)$ with $au = ua = 1$ and $ae = ea$ then $(eae)(eue) = (eue)(eae) = e$.

(3). We use the equality

$$U(eRe) = (eRe) \cap (\overline{e} + U(R)).$$

Assume $a \in U(eRe) \cap U(\overline{e}R\overline{e})$. Then, $a = \overline{e} + u \in eRe$ with $u \in U(R)$ so there exists $v \in R$ with $(a - \overline{e})v = v(a - \overline{e}) = 1$. By left and right multiplication (respectively) with e we get $eave = evae = e$. Similarly, from $a \in U(\overline{e}R\overline{e})$, for $a = e + w$ there exists $t \in R$ such that $(a - e)t = t(a - e)$. By left and right multiplication (respectively) with $\overline{e}$, we get $\overline{e}at\overline{e} = \overline{e}ta\overline{e} = \overline{e}$. Finally, by addition, $a(eve + \overline{e}t\overline{e}) = (eve + \overline{e}t\overline{e})a = e + \overline{e} = 1$.

Notr that the proof shows that

$$U(eRe) \cap U(\overline{e}R\overline{e}) \subseteq U(R),$$

for every idempotent e which commutes with the units of R. $\square$