Beitr Algebra Geom
https://doi.org/10.1007/s13366-024-00756-9

ORIGINAL PAPER

The formula $ABA = Tr(AB)A$ for matrices

Grigore Călugăreanu[1]

Received: 8 June 2024 / Accepted: 10 June 2024

Abstract
We prove that this formula characterizes the square matrices over commutative rings for which all 2×2 minors equal zero.

Keywords Matrix · Trace · 2×2 minors · Block multiplication

Mathematics Subject Classification 15B33 · 16S50

1 Introduction

An important notion in Ring Theory is obtained for every idempotent e of a ring R: the set of products eRe, called a *corner* of the ring R, which is readily seen to be a ring itself with e the multiplicative identity. The starting point of this note was a Ring Theory research in which elements of form eae with idempotent e and unit (or nilpotent) a occurred, that is, some specific elements in some corners of a ring. To have more examples, we tried to check what can be said about matrix products of this sort.

If one considers 2×2 matrices over commutative rings, it is readily checked (by direct computation) that if $\det(A) = 0$ and B is an arbitrary matrix then

$$ABA = Tr(AB)A$$

formula we did not find in the literature. A little bit harder, but still not difficult, the formula can also be checked (by direct computation) for 3×3 matrices, whenever all the 2×2 minors of A equal zero.

In this short note, we prove the formula for $n \times n$ matrices over commutative rings and any $n \geq 2$. Surprisingly, the converse also holds and so we have a characterization result.

✉ Grigore Călugăreanu
calu@math.ubbcluj.ro

1 Department of Mathematics, Babeş-Bolyai University, 1 Kogălniceanu Street, Cluj-Napoca, Romania

Published online: 23 June 2024

Theorem 1 *Let A, B be $n \times n$ matrices over a commutative ring R and $n \geq 2$. The formula*

$$ABA = Tr(AB)A = ATr(BA),$$

holds for every matrix B if and only if all 2×2 minors of A equal zero.

Notice that the second equality follows from the hypothesis that the base ring is commutative and the well-known trace formula $Tr(AB) = Tr(BA)$.

In Sect. 2 we give the proof of the characterization, by induction and using block multiplication one way, in the general commutative base ring case.

In Sect. 3 we present simple proofs in two special cases: when A has inner rank one (that is, has a column-row decomposition) and when the entries of A belong to a field, respectively.

The final Sect. 4 includes some comments and applications.

Recall the following

Definition The *inner rank* of an $n \times n$ matrix A over a ring R is the least integer r such that A can be expressed as a product of an $n \times r$ matrix and an $r \times n$ matrix. For example, over a division ring this notion coincides with the usual notion of rank. A square matrix is called *full* if its inner rank equals its order, and *non-full* otherwise.

It is easy to see that, over any commutative ring and for any positive integer $n \geq 2$, the determinant of an n-column-n-row product is zero. Obviously, such products have inner rank 1.

We also recall that a ring was called *pre-Schreier* (see (Zafrullah 1987)) if every element is *primal* (i.e., if r divides a product xy then $r = ab$ with $a \mid x$ and $b \mid y$).

2 Proof of the theorem

The formula is obviously true for $n = 1$ without any hypothesis. Indeed $aba = (ab)a = a(ba)$ holds over any (associative) ring (possibly not commutative nor unital).

Proof One way, using induction and block multiplication, we provide a proof of the formula for $n \times n$ matrices over commutative rings, whenever all 2×2 minors equal zero. As noticed in the Introduction, if $\det(A) = 0$ for a 2×2 matrix (over any commutative ring) $A = [a_{ij}]$ and $B = [b_{ij}]$ is arbitrary, it is not hard to check $ABA = Tr(AB)A$ (here $Tr(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}$).

Let $A, B \in \mathbb{M}_n(R)$ for a commutative ring R. In both A and B we emphasize the $(n-1) \times (n-1)$ left upper corner, that is, we write $A = \begin{bmatrix} A' & \mathbf{a}_2 \\ \mathbf{a}_1 & a_{nn} \end{bmatrix}$ with $A' \in \mathbb{M}_{n-1}(R)$, one $1 \times (n-1)$ row $\mathbf{a}_1 = \begin{bmatrix} a_{n1} & \ldots & a_{n,n-1} \end{bmatrix}$ and one $(n-1) \times 1$ column $\mathbf{a}_2 = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{n-1,n} \end{bmatrix}$.

We use similar notations for B and block multiplication for these matrices.

Then $AB = \begin{bmatrix} A'B' + \mathbf{a}_2\mathbf{b}_1 & A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn} \\ \mathbf{a}_1 B' + a_{nn}\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn} \end{bmatrix}$ and $ABA =$

$$\begin{bmatrix} (A'B' + \mathbf{a}_2\mathbf{b}_1)A' + (A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn})\mathbf{a}_1 & (A'B' + \mathbf{a}_2\mathbf{b}_1)\mathbf{a}_2 + (A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn})a_{nn} \\ (\mathbf{a}_1 B' + a_{nn}\mathbf{b}_1)A' + (\mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn})\mathbf{a}_1 & (\mathbf{a}_1 B' + a_{nn}\mathbf{b}_1)\mathbf{a}_2 + (\mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn})a_{nn} \end{bmatrix}.$$

By induction hypothesis, assume $A'B'A' = Tr(A'B')A'$. In order to prove $ABA = Tr(AB)A = Tr(AB)\begin{bmatrix} A' & \mathbf{a}_2 \\ \mathbf{a}_1 & a_{nn} \end{bmatrix}$ we have to check four equalities:

(i) $Tr(A'B')A' + \mathbf{a}_2\mathbf{b}_1 A' + (A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn})\mathbf{a}_1 = Tr(AB)A'$
(ii) $(A'B' + \mathbf{a}_2\mathbf{b}_1)\mathbf{a}_2 + (A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn})a_{nn} = Tr(AB)\mathbf{a}_2$
(iii) $(\mathbf{a}_1 B' + a_{nn}\mathbf{b}_1)A' + (\mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn})\mathbf{a}_1 = Tr(AB)\mathbf{a}_1$
(iv) $(\mathbf{a}_1 B' + a_{nn}\mathbf{b}_1)\mathbf{a}_2 + (\mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn})a_{nn} = Tr(AB)a_{nn}$.

Since $Tr(AB) = Tr(A'B' + \mathbf{a}_2\mathbf{b}_1) + \mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn}$, and $Tr(\mathbf{a}_2\mathbf{b}_1) = Tr(\mathbf{b}_1\mathbf{a}_2) = \mathbf{b}_1\mathbf{a}_2$, the equalities amount to

(i) $\mathbf{a}_2\mathbf{b}_1 A' + (A'\mathbf{b}_2 + \mathbf{a}_2 b_{nn})\mathbf{a}_1 = (\mathbf{b}_1\mathbf{a}_2 + \mathbf{a}_1\mathbf{b}_2 + a_{nn}b_{nn})A'$
(ii) $(A'B' + \mathbf{a}_2\mathbf{b}_1)\mathbf{a}_2 + A'\mathbf{b}_2 a_{nn} = (Tr(A'B') + \mathbf{b}_1\mathbf{a}_2 + \mathbf{a}_2\mathbf{b}_1)\mathbf{a}_2$
(iii) $(\mathbf{a}_1 B' + a_{nn}\mathbf{b}_1)A' = (Tr(A'B') + \mathbf{b}_1\mathbf{a}_2)\mathbf{a}_1$
(iv) $\mathbf{a}_1 B'\mathbf{a}_2 = Tr(A'B')a_{nn}$.

We just provide some details of how all these equalities are verified.

(i) The LHS has a sum of three $(n-1) \times (n-1)$ matrices and the RHS has A' multiplied by a scalar. We have to check the equalities of the entries in the LHS matrix respectively in the RHS matrix. Each entry is a linear combination of b's with coefficients products of two a's. Some corresponding entries are already equal (e.g., coefficients of b_{n1} in the corresponding $(1, n-1)$ entries), the other use the vanishing of the 2×2 minors (e.g., we use $\begin{vmatrix} a_{n-1,1} & a_{n-1,n-1} \\ a_{n1} & a_{n,n-1} \end{vmatrix} = 0$ in order to check the equality of the products which multiply $b_{n-1,n}$, for the $(n-1, 1)$ entries).
(ii) The LHS is a sum of three columns and the RHS is a product of a scalar and a column. To check the equality amounts to verify the equality of the corresponding $n-1$ entries. As in the previous case, each entry is a linear combination of b's with coefficients products of two a's. Some corresponding entries are already equal (e.g., coefficients of b_{n1} in the corresponding upper entries), the other use the vanishing of the 2×2 minors (e.g., we use $\begin{vmatrix} a_{11} & a_{1n} \\ a_{n-1,1} & a_{n-1,n} \end{vmatrix} = 0$ in order to check the equality of the products which multiply $b_{1,n-1}$, for the upper entries).
(iii) Both the LHS and RHS of the equality are linear combination of all b_{ij} for $1 \le i \le n$, $1 \le j \le n-1$ with coefficients products of two a's. For every pair (i, j), the equality of the coefficients of b_{ij} in both sides amounts to the vanishing of a minor of type $\begin{vmatrix} a_{ji} & a_{jj} \\ a_{ni} & a_{nj} \end{vmatrix}$ or $\begin{vmatrix} a_{j1} & a_{ji} \\ a_{n1} & a_{ni} \end{vmatrix}$ or similar.
(iv) Both the LHS and RHS of the equality are linear combination of all b_{ij} for $1 \le i, j \le n-1$ with coefficients products of two a's. For every pair $1 \le i, j \le n-1$, the equality of the coefficients of b_{ij} in both sides amounts precisely to the vanishing of the minor $\begin{vmatrix} a_{ji} & a_{jn} \\ a_{ni} & a_{nn} \end{vmatrix}$.

As for the converse, take an arbitrary minor $m = \begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix}$ and choose $B = E_{lj}$. It is easy to see that $C =: AE_{lj}A = \mathrm{col}_l(A) \cdot \mathrm{row}_j(A)$ and since $Tr(AE_{lj}) = a_{jl}$,

$D =: Tr(AE_{lj})A = a_{jl}A$. Now compute $c_{ik} = a_{jk}a_{il}$ and $d_{ik} = a_{jl}a_{ik}$. Since $C = D$, the minor m indeed equals zero. □

Remark For $n = 2$, the converse (that is, $\det(A) = 0$) follows easily by the Cayley-Hamilton theorem, taking $B = I_2$. In the $n = 3$ case, the same choice, $B = I_3$, and the Cayley-Hamilton theorem still gives $\det(A) = 0$ (since $A^2 = Tr(A)A$ and $Tr(A^2) = Tr^2(A)$) but not the vanishing of all the 2×2 minors.

3 Special cases with simple proofs

It is easy to prove our formula **if** $A = \mathbf{c} \cdot \mathbf{r}$ **has a column-row decomposition** (i.e., the inner rank of A is 1, the matrix is *non-full*) over any commutative ring.

Proof We use the *cyclicity of the trace*, i.e., the trace is invariant with respect to cyclic permutations of its argument. For example, $Tr(ABC) = Tr(BCA) = Tr(CAB)$. Let $\mathbf{c}$ be an $n \times 1$ column and let $\mathbf{r}$ be an $1 \times n$ row such that $A = \mathbf{c} \cdot \mathbf{r}$. Then $Tr(AB) = Tr(\mathbf{c} \cdot (\mathbf{r} \cdot B)) = Tr(\mathbf{r} \cdot B \cdot \mathbf{c}) = \mathbf{r} \cdot B \cdot \mathbf{c}$, since this is a scalar. Therefore $ABA = \mathbf{c} \cdot \mathbf{r} \cdot B \cdot \mathbf{c} \cdot \mathbf{r} = \mathbf{c}Tr(AB)\mathbf{r} = Tr(AB)A$. □

However, having a column-row decomposition for any matrix with all 2×2 minors equal to zero, requires additional conditions on the base ring.

For instance, in the $n = 2$ case, it is proved (see (Călugăreanu and Pop 2021)) that the existence of a column-row decomposition for a zero determinant 2×2 matrix is equivalent to the base ring being a pre-Schreier domain.

As it is well-known, the proof is valid if the base ring is a field, as, for any matrix, the inner rank equals the rank.

Next we supply **a different proof for** (square) **matrices over fields**.

First we mention two facts which are easier to describe for linear maps (instead of matrices).

Suppose f is a linear map from V to itself, where V is some finite n -dimensional vector space over a field K. Recall that, by definition, the rank of f is $r = \dim(f(V))$.

1. Suppose $r = 1$. Then the range $f(V)$ has a vector basis, say $\{\mathbf{v}\}$ for some $\mathbf{v} \neq \mathbf{0}$. Since for every $\mathbf{x} \in V$, $f(\mathbf{x}) \in f(V)$ it follows that $f(\mathbf{x}) = d\mathbf{v}$ for some $d \in K$. In particular $f(\mathbf{v}) = c\mathbf{v}$ for some $c \in K$, so $\mathbf{v}$ is an eigenvector corresponding to the eigenvalue c.

2. Suppose $r = 1$ and so the nullity $\dim(\ker(f)) = n - 1$. Since the multiplicity of an eigenvalue is at least the dimension of the corresponding eigenspace, it follows that 0 is an eigenvalue with multiplicity at least $n - 1$. Moreover, as the sum of all eigenvalues (counted with multiplicity) is $Tr(f)$, the last eigenvalue is $Tr(f)$.

Now, given some square matrix A, we can apply this to the map canonically attached to A. Since the formula trivially holds for $B = 0$, we assume $B \neq 0$.

Proof Suppose $\text{rank}(A) = 1$ (and so $A \neq 0$). By the above (see **1**), there is a nonzero vector $\mathbf{v}$ such that for each vector $\mathbf{x}$, $A\mathbf{x} = d\mathbf{v}$ for some $d \in R$. Then we compute

$$ABA\mathbf{x} = AB(d\mathbf{v}) = dA(B\mathbf{v}) = dc\mathbf{v} = c(d\mathbf{v}) = cA\mathbf{x}$$

for some $c \in R$ (that is, $A(B\mathbf{v}) = c\mathbf{v}$). Since the equality holds for every $\mathbf{x}$, $ABA = cA$ follows.

Finally, it remains to notice that $\text{rank}(AB) = 1$ and since $(AB)\mathbf{v} = c\mathbf{v}$ we deduce (see **2** above) that $c = Tr(AB)$. $\square$

Despite the fact that notions like rank of a matrix, linearly independent vector (or column in R^n), null space, eigenvalue and eigenvector can be defined for matrices over commutative rings (see (Brown 1993)), this proof cannot be extended to the general case of matrices over commutative rings. There are several impediments occurring.

One of these is that if $c \in R$ is an eigenvalue (e.g., a root of the characteristic polynomial $p_A(X)$) it does not follow that the eigenvector of A associated to c is independent over R. Of course, $p_A(X)$ may not have roots in R.

Moreover, $\text{rank}(A) = 1$ implies that the 2×2 minors are zero, but not conversely. Unlike the field case, a matrix can have rank zero without being the zero matrix.

4 Comments and applications

Two special cases of this formula can be found on MathOverflow.

(1) Let A be an $n \times n$ complex matrix having rank 1. Prove that $A^2 = cA$ for some scalar c (see van Leeuwen 2014, solution by M. van Leeuwen 2014), and
(2) Show that if $Tr(AB) = 0$ and A has rank 1 then $ABA = 0$ (see EuYu 2016, solution by EuYu 2016).

The first follows from our formula by taking $B = I_n$, where c turns out to be precisely $Tr(AB)$, and the second follows directly from the formula, whenever $Tr(AB) = 0$.

However, as our theorem proves, both hold over any commutative ring, not only for complex matrices, or over special integral domains (where a column-row splitting is possible).

An easy but general case which was not mentioned follows from Cayley–Hamilton theorem: let A be a zero determinant 2×2 matrix over any commutative ring. Then $A^2 = Tr(A)A$.

The following applications are straightforward.

Corollary 2 *Assume $n \geq 2$, all 2×2 minors of the $n \times n$ matrix A equal zero and the $n \times n$ matrix B is arbitrary. Then*

(a) $(AB)^2 = Tr(AB)AB$ *but not conversely, unless B is a unit.*
(b) $Tr(ABA) = Tr(AB)Tr(A) = Tr(A)Tr(BA)$; *in particular* $Tr(A^2) = Tr^2(A)$.

It is well-known that if R is any commutative ring, the endomorphisms of an R-module M form an algebra over R denoted $End_R(M)$. Then there is a canonical R-linear map: $M^* \otimes_R M \longrightarrow R$ induced through linearity by $f \otimes x \mapsto f(x)$; it is the unique R-linear map corresponding to the natural pairing. If M is a finitely generated projective R-module, then one can identify $M^* \otimes_R M = End_R(M)$ through the canonical homomorphism mentioned above and then the above is the trace map: $Tr : End_R(M) \longrightarrow R$. Our formula can be transferred *mutatis mutandis* to this context, characterizing endomorphisms $f \in End_R(M)$ with $rank(f(M)) = 1$ (equivalently, $fgf = Tr(fg)f$ for every $g \in End_R(M)$).

Remark A similar formula fails for matrices over *noncommutative* rings, even over (noncommutative) division rings, replacing the zero determinant hypothesis with the rank 1 condition.

Example Over the real quaternions take $ABA = \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ \mathbf{j} & \mathbf{j} \end{bmatrix} \begin{bmatrix} \mathbf{j} & \mathbf{j} \\ \mathbf{k} & \mathbf{k} \end{bmatrix} \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ \mathbf{j} & \mathbf{j} \end{bmatrix} = \begin{bmatrix} \mathbf{k}-\mathbf{j} & \mathbf{k}-\mathbf{j} \\ \mathbf{i}-1 & \mathbf{i}-1 \end{bmatrix} \begin{bmatrix} \mathbf{i} & \mathbf{i} \\ \mathbf{j} & \mathbf{j} \end{bmatrix} = \begin{bmatrix} \mathbf{j}+\mathbf{k}-\mathbf{i}+1 & * \\ * & * \end{bmatrix}$. However, $Tr(AB) = \mathbf{k} - \mathbf{j} + \mathbf{i} - 1$ and the upper left entry of $Tr(AB)A$ is $\mathbf{j} + \mathbf{k} - \mathbf{i} - 1$.

Since our paper is entitled "the formula $ABA = Tr(AB)B$ for matrices", one could wonder which are the matrices B such that the formula holds for every A.

It is easy to prove the following

Proposition 3 *For $n \times n$ matrices A, B over any commutative ring, the formula $ABA = Tr(AB)A$ holds for every A if and only if $B = 0_n$.*

Proof Taking $A = I_n$ gives $B = Tr(B)I_n$ so B is a scalar matrix, say $B = tI_n$ with $t = Tr(B)$. Thus the formula becomes $tA^2 = tTr(A)A$. Now take A any invertible matrix with zero trace (e.g. $\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ if n is odd and remove last row and last column, if n is even). Then $tA^2 = 0_n$ and so $t = 0$. □

Declarations

Conflict of interest The author reports there are no conflict of interest to declare.

References

Brown, W.C.: Matrices Over Commutative Rings. Marcel Dekker Inc., New York (1993)
Călugăreanu, G., Pop, H.F.: On zero determinant matrices that are full. Math. Panon. **27**(2), 81–88 (2021)

EuYu https://math.stackexchange.com/questions/1787389/show-that-if-operatornametraceab-0-and-operatornamerank-a-1-then (2016)
van Leeuwen, M.: https://math.stackexchange.com/questions/991346/matrix-of-rank-1-show-that-a2-ca-for-some-scalar-c (2014)
Zafrullah, M.: On a property of pre-Schreier domains. Commun. Algebra **15**(9), 1895–1920 (1987)