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Unipotent similarity for matrices over commutative domains

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Abstract. A unit u of a ring is called unipotent if $u-1$ is nilpotent. We characterize the similarity of 2×2 matrices over commutative domains, realized by unipotent matrices, i.e., $B = U^{-1}AU$ with unipotent matrix U.

1 Introduction

In this note R denotes an associative ring with identity (for short, *unital* ring), $U(R)$ the group of units, $N(R)$ the set of nilpotent elements and $\mathbb{M}_n(R)$ the corresponding matrix ring (i.e., the set of all $n \times n$ matrices with entries in R). An element u of a ring is called *unipotent* if $u-1$ is nilpotent. That is, $u = 1+t$ for some nilpotent t. Over any (unital) ring it is easy to check that unipotents are units.

Two elements $a, b \in R$ are called *conjugate* if there is a unit $u \in U(R)$ such that $b = u^{-1}au$. Two square matrices $A, B \in \mathbb{M}_n(R)$ are called *similar* if these are conjugate in the matrix ring $\mathbb{M}_n(R)$. In the sequel we consider the following

Definition. Two elements a, b of a ring R are *unipotent conjugate* if there is a unipotent u such that $b = u^{-1}au$, that is, if there is a nilpotent $t \in N(R)$ such that $b = (1+t)^{-1}a(1+t)$.

Clearly, unipotent conjugate elements are conjugate. Examples will show that the converse fails, even for special classes of elements (i.e., idempotents,

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nilpotents or units). Two square matrices $A, B \in \mathbb{M}_n(R)$ will be called *unipotent similar* if these are unipotent conjugate in the matrix ring $\mathbb{M}_n(R)$.

Our goal in this note is to find a *criterion* (Theorem 1) for two matrices in order to be unipotent similar. According to the above definition, two (square) matrices A, B are unipotent similar if we can find a unipotent matrix U such that $AU = UB$, or equivalently, a nilpotent matrix T, such that $A(I_n + T) = (I_n + T)B$. If we denote the entries of T by t_{ij}, $1 \leq i, j \leq n$, it is readily seen that this equality amounts to a linear system in the unknown entries of T. Hence, if the base ring R is a field, our problem is easy to solve using basic linear algebra. The divisibility relations in the characterization provided by Proposition 1, are no longer an issue.

However, we want to find a more general environment (that is, a larger class of rings) in which we still can use some basic linear algebra methods.

The first necessary restriction, in order to be able to use *determinants*, is that we suppose the ring R is commutative. The reader can use the excellent book of William K. Brown, "Matrices over commutative rings" in order to have a complete look of what remains true when passing from fields to arbitrary commutative rings (including a suitable notion of rank, solving linear systems of equations etc).

The second necessary restriction, in order to have a *known form of the nilpotent matrices*, is that we suppose the commutative ring R to be a domain (i.e., an integral domain). For $n = 2$, a matrix is nilpotent if and only if it has zero determinant and zero trace. For $n \geq 3$ there are conditions which characterize the nilpotent matrices, but more complicated (e.g., see [2]).

As mentioned in [1] (**4.13**), if R is a commutative domain with quotient field F, the rank of a matrix A over R (see first paragraph of Section 2) is just the classical rank of A when A is viewed as a matrix over F. Thus, when solving a linear system of equations over R, we can solve this over F and then find the conditions which assure the solution belongs to R. Of course, over F, we can use the Kronecker (Rouché) - Capelli theorem and Cramer's rule too.

In this note we describe the unipotent similarity for 2×2 matrices over (commutative) domains. As customarily, $[A, \mathbf{b}]$ denotes the augmented matrix.

2 The 2×2 matrix unipotent similarity

For a commutative ring R and any positive integer m, the ideal $D_m(A)$ of R generated by the $m \times m$ minors of a matrix A was called the m-*th determinantal ideal* of A and we put $D_0(A) = R$. These are used in order to define a rank notion for matrices over any commutative ring (the analogue of the maximum order of nonzero minors). Namely (see [1], chapter 4), these ideals form an ascending sequence of ideals

$$(0) = D_{n+1}(A) \subseteq D_n(A) \subseteq \ldots \subseteq D_1(A) \subseteq D_0(A) = R$$

and the *rank of* A is $\mathrm{rk}(A) := \max\{m : \mathrm{ann}_R(D_m(A)) = (0)\}$. Here, for an ideal I of R, $\mathrm{ann}_R(I)$ is the *annihilator* of I, that is, $\{a \in R : aI = (0)\}$.

As already mentioned in the introduction, if F is the quotient field of R and $A \in \mathbb{M}_n(R)$ then $\mathrm{rk}(A) = \mathrm{rank}_F(A)$.

Let R be a commutative domain, $A, B \in \mathbb{M}_2(R)$ and let T be a nilpotent 2×2 matrix. Then $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ with $x^2 + yz = 0$ (that is, has zero trace and zero determinant) and A, B are unipotent similar if and only if there is a matrix T of the previous form such that $A(I_2 + T) = (I_2 + T)B$. We denote $A = [a_{ij}]$, $B = [b_{ij}]$. Since unipotent similar matrices are similar and so have the same determinant and the same trace, we first prove the following

Proposition 1 *Let* R *be a commutative domain and let* $A, B \in \mathbb{M}_2(R)$ *be such that* $\det(A) = \det(B)$ *and* $\mathrm{Tr}(A) = \mathrm{Tr}(B)$. *There exists a zero trace matrix* T *such that* $A(I_2 + T) = (I_2 + T)B$ *if and only if any of the following three conditions is fulfilled*

(i) *there exists* z *such that* $a_{21} + b_{21}$ *divides* $b_{21} - a_{21} - z(b_{22} - a_{11})$ *and* $2(a_{11} - b_{11}) + z(a_{12} + b_{12})$;

(ii) *there exists* y *such that* $a_{12} + b_{12}$ *divides* $a_{12} - b_{12} - y(b_{22} - a_{11})$ *and* $2(b_{11} - a_{11}) + y(a_{21} + b_{21})$;

(iii) *there exists* x *such that* $b_{22} - a_{11}$ *divides* $a_{12} - b_{12} - x(a_{12} + b_{12})$ *and* $b_{21} - a_{21} - x(a_{21} + b_{21})$.

Proof. We start with an unknown zero trace matrix $T = \begin{bmatrix} x & y \\ z & -x \end{bmatrix}$ and write $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} = \begin{bmatrix} 1+x & y \\ z & 1-x \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. This equality is equivalent to a linear system of 4 equations and 3 unknowns which we write

$$MX = \begin{bmatrix} a_{11} - b_{11} & -b_{21} & a_{12} \\ a_{12} + b_{12} & b_{22} - a_{11} & 0 \\ a_{21} + b_{21} & 0 & a_{22} - b_{11} \\ a_{22} - b_{22} & -a_{21} & b_{12} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_{11} - a_{11} \\ a_{12} - b_{12} \\ b_{21} - a_{21} \\ a_{22} - b_{22} \end{bmatrix} = N.$$

We consider this system over the quotient field F. Using $\det(A) = a_{11}a_{22} - a_{12}a_{21} = b_{11}b_{22} - b_{12}b_{21} = \det(B)$ and $\mathrm{Tr}(A) = a_{11} + a_{22} = b_{11} + b_{22} = \mathrm{Tr}(B)$, it can be shown that the system matrix M and the augmented matrix

$$[M|N] = \left[\begin{array}{ccc} a_{11} - b_{11} & -b_{21} & a_{12} \\ a_{12} + b_{12} & b_{22} - a_{11} & 0 \\ a_{21} + b_{21} & 0 & a_{22} - b_{11} \\ a_{22} - b_{22} & -a_{21} & b_{12} \end{array} \left[\begin{array}{c} b_{11} - a_{11} \\ a_{12} - b_{12} \\ b_{21} - a_{21} \\ a_{22} - b_{22} \end{array}\right]\right],$$

both have rank 2.

For M we have just four 3×3 minors to check and for $[M|N]$ we have another twelve 3×3 minors to check. We skip the easy calculations.

According to Kronecker (Rouché) - Capelli theorem, the system is solvable and using Cramer's rule we choose an independent unknown and solve the system for the other two dependent unknowns. The initial linear system is equivalent to any two independent equations.

For instance, by Cramer's rule, for (i) we choose the first two equations

$$\begin{array}{rcl} (a_{11} - b_{11})x - b_{21}y & = & b_{11} - a_{11} - a_{12}z \\ (a_{12} + b_{12})x + (b_{22} - a_{11})y & = & a_{12} - b_{12} \end{array}.$$

By elimination we get $x\Delta = \Delta_x$, $y\Delta = \Delta_y$, with the determinant

$\Delta = \det \begin{bmatrix} a_{11} - b_{11} & -b_{21} \\ a_{12} + b_{12} & b_{22} - a_{11} \end{bmatrix} = a_{12}(a_{21} + b_{21})$,

$\Delta_x = \det \begin{bmatrix} b_{11} - a_{11} - a_{12}z & -b_{21} \\ a_{12} - b_{12} & b_{22} - a_{11} \end{bmatrix} = a_{12}[b_{21} - a_{21} - z(b_{22} - a_{11})]$ and

$\Delta_y = \det \begin{bmatrix} a_{11} - b_{11} & b_{11} - a_{11} - a_{12}z \\ a_{12} + b_{12} & a_{12} - b_{12} \end{bmatrix} = a_{12}[2(a_{11} - b_{11}) + z(a_{12} + b_{12})]$.

That is, if $\Delta \neq 0$, the system is equivalent to

$$\begin{array}{rcl} a_{12}(a_{21} + b_{21})x & = & a_{12}[b_{21} - a_{21} - z(b_{22} - a_{11})] \\ a_{12}(a_{21} + b_{21})y & = & a_{12}[2(a_{11} - b_{11}) + z(a_{12} + b_{12})] \end{array}.$$

If $a_{12} \neq 0$, by cancellation a solution (x, y, z) exists iff the condition (i) holds.

If $a_{12} = 0$ the (initial) system has the solution $x = -1$, $y = 0$ and $z = \dfrac{2(b_{11} - a_{11})}{b_{12}} = \dfrac{2b_{21}}{b_{22} - a_{11}}$ iff b_{12} divides $2(b_{11} - a_{11})$ or equivalently, $b_{22} - a_{11}$ divides $2b_{21}$.

Choosing other pairs of independent equations from the (initial) system we obtain the conditions (ii) and (iii), respectively. □

Remarks. 1) If all $a_{21}+b_{21}=a_{12}+b_{12}=b_{22}-a_{11}=0$ then $B=adj(A)$, the adjugate. The conditions show that $A=B$ are diagonal with equal entries on the diagonal, i.e., $A=B=a_{11}I_2$, obviously unipotent similar.

2) In particular, if any of the divisibilities below hold, we can choose $z=0$ (resp. $y=0$ resp. $x=0$) for a solution (x,y,z).

(i) $a_{21}+b_{21}$ divides both $b_{21}-a_{21}$ and $2(a_{11}-b_{11})$;

(ii) $a_{12}+b_{12}$ divides both $b_{12}-a_{12}$ and $2(a_{11}-b_{11})$;

(iii) $b_{22}-a_{11}$ divides both $a_{12}-b_{12}$ and $b_{21}-a_{21}$.

Formally using fractions, accordingly, we have

(i) $x=\dfrac{b_{21}-a_{21}}{a_{21}+b_{21}}, y=\dfrac{2(a_{11}-b_{11})}{a_{21}+b_{21}}$, or

(ii) $x=\dfrac{a_{12}-b_{12}}{a_{12}+b_{12}}, z=\dfrac{2(b_{11}-a_{11})}{a_{12}+b_{12}}$, or

(iii) $y=\dfrac{a_{12}-b_{12}}{b_{22}-a_{11}}, z=\dfrac{b_{21}-a_{21}}{b_{22}-a_{11}}$.

Only one more condition is necessary in order to describe the unipotent similarity for 2×2 matrices over commutative domains.

Theorem 1 *Let R be a commutative domain and let $A,B\in\mathbb{M}_2(R)$ be such that $\det(A)=\det(B)$ and $\mathrm{Tr}(A)=\mathrm{Tr}(B)$. The matrices A, B are unipotent similar if and only if any of the conditions (i), (ii) or (iii) in Proposition 1 holds and for the corresponding solution, the quadratic equation in z (resp. y or x) $x^2+yz=0$ is solvable. Accordingly, any of the corresponding quadratic equations should be solvable*

(i) $[b_{21}-a_{21}-z(b_{22}-a_{11})]^2+[2(a_{11}-b_{11})+z(a_{12}+b_{12})](a_{21}+b_{21})z=0$, *or*

(ii) $[a_{12}-b_{12}-y(b_{22}-a_{11})]^2+[2(b_{11}-a_{11})+y(a_{21}+b_{21})](a_{21}+b_{21})y=0$, *or*

(iii) $(b_{22}-a_{11})^2x^2-[a_{12}-b_{12}-x(a_{12}+b_{12})][a_{21}-b_{21}+x(a_{21}+b_{21})]=0$.

3 Examples

In this section, using the characterization proved in the previous section, we mainly give examples of *similar matrices which are not unipotent similar*. The examples are over the integers. Among these examples we choose idempotents, nilpotents and units. In the next five examples, to simplify the exposition, the *similarity* of the pair of 2×2 matrices is given by $U = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, with $U^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Actually, we check the solvability of the quadratic equations (Theorem 1) on the examples below.

(1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = U^{-1}AU = \begin{bmatrix} 6 & -2 \\ 2 & -1 \end{bmatrix}$.

The linear system reduces to $5x = 2z - 1$, $y = -2$ and so (for example (i); equivalently, (ii) or (iii)) $25(x^2 + yz) = (2z - 1)^2 - 50z = 0$ has *no* integer solutions. According to Theorem 1, *these similar matrices are not unipotent similar over* $\mathbb{Z}$. Here A, B have no special property: $\det(A) = \det(B) = -2$, $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 5$.

(2) $E = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, $F = U^{-1}EU = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$.

The linear system (for example (ii)) reduces to $x = 3 + 2y$, $z = 2 + 2y$ and $x^2 + yz = (3y+2)^2 + 2y(y+1) = 11y^2 + 14y + 4 = 0$ with *no* rational solutions.

According to Theorem 1, *these similar idempotents (indeed, zero determinants and traces* $= 1$*) are not unipotent similar over* $\mathbb{Z}$.

(3) $N = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $N_1 = U^{-1}NU = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$.

The linear system (for example (iii)) reduces to $y = -2$, $z = -1 + x$ and $x^2 + yz = x^2 - 2x + 2 = (x - 1)^2 + 1 = 0$ with *no* real solutions. According to Theorem 1, *these similar nilpotents (indeed, zero determinants and zero traces) are not unipotent similar over* $\mathbb{Z}$.

(4) $V = E_{12} + E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $V_1 = U^{-1}VU = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$.

The linear system (for example (i)) reduces to $2x = z$, $2y = -2 + z$ and $4(x^2 + yz) = z(3z - 4) = 0$, which, over any integral domain where 3 is not a unit (e.g., over $\mathbb{Z}$), has only the solution $z = 0$. Accordingly $x = 0$ and $y = -1$ and indeed, for $T = -E_{12}$, that is $I_2 + T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, $V(I_2 + T) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = (I_2 + T)V_1$.

Therefore (as the matrix equality $V(I_2+T) = (I_2+T)V_1$, recorded in (i) holds over any unital ring) *these two similar units (indeed, determinants* $= -1$*) are also unipotent similar over any (unital) ring.*

(5) $W = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, $W_1 = U^{-1}WU = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$.

The linear system (for example (ii)) reduces to $3x = 2y + 1$, $z = y$ and $9(x^2 + yz) = (2y + 1)^2 + 9y^2 = 0$ has *no* real solutions. According to Theorem 1, *these similar units* (*indeed, determinants* $= 1$) *are not unipotent similar over* $\mathbb{Z}$.

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