1-SYLVESTER MATRICES

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ABSTRACT. A nonzero element a is called *1-Sylvester* in a ring R, if there exist $b, c \in R$ such that $1 = ab + ca$. In this paper we study such elements, mainly in matrix rings over commutative rings. In particular, we study the case when $b = c$, when b is called an *anticommutator inverse* for a.

1. INTRODUCTION

The Sylvester equation is a classic matrix equation of the form $AX + XB = C$, where A, B, and C are known matrices, and X is the unknown. Named after James Joseph Sylvester, this equation became central in areas like control theory, linear differential equations, and numerical linear algebra.

This paper focuses on a special case of the Sylvester equation when $C = I_n$ and particularly when $B = C$, in the context of matrices over commutative rings. We introduce the notion of 1-Sylvester elements in a ring R with identity: a nonzero $a \in R$ is *1-Sylvester* if there exist $b, c \in R$ such that $1 = ab + ca$.

Although (the so-called) weakly fadelian rings (those where every nonzero element is 1-Sylvester, see [4]) are simple domains, matrix rings are not domains. Thus, studying 1-Sylvester elements in matrix rings requires different approaches. A special case involves anticommutators: for $a, b \in R$, $[a, b]_+ = ab + ba$. If $[a, b]_+ = 1$, then b will be called an *anticommutator inverse* (ACI) of a.

Our study of 1-Sylvester elements and ACIs is motivated in part by their relevance in quantum mechanics, where anticommutation relations govern fermionic operators. Not all elements have an anticommutator inverse; some have none, others have one or many.

Finally, we note that while weakly fadelian rings are domains, individual 1-Sylvester elements may be zero divisors or nilpotent. For example, $E_{12} \in \mathbb{M}_2(R)$ satisfies $E_{12}E_{21} + E_{21}E_{12} = I_2$ but is nilpotent. Here E_{ij} denotes the $n \times n$ matrix with all entries zero, excepting the (i, j)-entry which is 1.

Throughout this paper, all rings are assumed to be associative, unital, and nonzero (i.e., $1 \neq 0$). For matrix rings, unless otherwise stated, we assume that the base ring is commutative. For a ring R, $U(R)$ denotes the set of all units of R. Two elements a, b in a commutative ring R are called *coprime* if there exist $c, d \in R$ such that $ca + db = 1$. As is customary, $|$ denotes the binary relation of divisibility.

This paper is structured as follows: Section 2 begins by discussing general properties of 1-Sylvester elements in arbitrary rings including the fact that in any ring,

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the only 1-Sylvester idempotent is the identity. We then prove that diagonal 1-Sylvester $n \times n$ matrices are precisely those that are invertible. Additionally, we show that numerous nilpotent 1-Sylvester $n \times n$ matrices also exist.

Section 3 focuses on characterizing upper triangular 1-Sylvester 2×2 matrices over commutative domains, including special cases such as ACI matrices and integral matrices. We also explore the potential uniqueness of the anticommutator inverse for 2×2 matrices over commutative rings. Simplifications are provided using the Kronecker product of matrices together with matrix vectorization.

2. 1-Sylvester $n \times n$ matrices

We begin by collecting some straightforward yet useful properties of 1-Sylvester elements in any (unital) ring. Two elements a, b in a ring R are called *equivalent* if there exist units p, q in R such that $b = paq$.

Proposition 2.1. *(i) The one-sided invertible elements are 1-Sylvester in any ring. However, units may not have anticommutator inverses.*

(ii) In a ring R, if 2 is a unit then all units have anticommutator inverses, while if 2 is not a unit, no central units have anticommutator inverses. However, a noncentral unit can have an anticommutator inverse.

(iii) The central 1-Sylvester elements of a ring R ring are precisely the units of R. In particular, the only 1-Sylvester elements of a commutative ring are the units.

(iv) The only 1-Sylvester idempotent is 1.

(v) Zero-square 1-Sylvester elements are (von Neumann) regular.

(vi) The 1-Sylvester property is preserved by (anti-)isomorphisms of rings. In particular, the 1-Sylvester property is invariant under conjugation.

(vii) If a is 1-Sylvester then ua is also 1-Sylvester, for every central unit u. In particular, $-a$ is also 1-Sylvester.

(viii) The 1-Sylvester property is not invariant under equivalences.

Proof. (i) Indeed, $1 = uv + 0 \cdot u$ takes care of right invertible elements and $1 = u \cdot 0 + wu$, of the left invertible ones. An example of 2×2 invertible matrix that has no anticommutator inverse is given in Remark 1, after Proposition 3.4.

(ii) If 2 is invertible then $(2u)^{-1}$ is an anticommutator inverse for the unit u. In $M_2(R)$, the noncentral unit $E_{12} + E_{21}$has the anticommutator inverse E_{12}.

(iv) Suppose $e^2 = e$ and $eb + ce = 1$ for some b, c. We multiply that equation on the left by $1 - e$ and then by $1 - e$ on the right. It follows that $1 - e = 0$, so $e = 1$.

(v) If $tb + ct = 1$ for $t^2 = 0$, multiplying the equation by t gives $tbt = tct = t$.

(vii) If $ab + ca = 1$ then for any central unit u, $(au)(u^{-1}c) + (cu^{-1})(ua) = 1$.

(viii) In $\mathbb{M}_2(R)$ over any ring R, the nilpotent E_{12} is 1-Sylvester (see Introduction), but the idempotent $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} E_{12} = E_{22}$ is not 1-Sylvester (see Proposition 2.3). □

Remarks. 1) In many rings (e.g. integral quaternions), the only 1-Sylvester elements are the units. For example, this is the case whenever the two-sided ideal generated by every non-unit is proper. This holds in rings that are not simple or in domains. Otherwise, in simple rings (like matrix rings, the focus of our paper), even non-units can generate the whole ring as a two-sided ideal. Actually, if a is 1-Sylvester in a ring R then $RaR = R$ (as $r = r \cdot 1 = r(ab + ca) = (ra)b + (rc)a$), but the converse fails (e.g., E_{11} in $\mathbb{M}_2(\mathbb{Z})$).

2) By a result of P. Ara (see [1]), it follows that the zero-square 1-Sylvester elements of exchange rings are even unit-regular.

According to our definition, an $n \times n$ matrix A over a ring R is 1-Sylvester if there are $B, C \in \mathbb{M}_n(R)$ such that $I_n = AB + CA$.

As witnessed by Proposition 2.1 (i), all one-sided invertible matrices are 1-Sylvester. Also from the previous proposition (vi) and (vii), recall that the 1-Sylvester property for matrices is *invariant under similarity* (and *under negatives*). Moreover, 1-Sylvester (square) matrices are *invariant under transpose.* In particular, if B is an anticommutator inverse of A, then B^T is an anticommutator inverse of A^T.

Next, we describe the diagonal 1-Sylvester matrices.

Proposition 2.2. *Over any commutative ring, the diagonal 1-Sylvester matrices are precisely the invertible diagonal matrices.*

Proof. Let $D = diag(d_1, ..., d_n)$ be a diagonal 1-Sylvester matrix. Suppose $DB + CD = I_n$ for some $n \times n$ matrices B, C. We just emphasize the diagonal of $DB + CD$: it is $diag(d_1(b_{11} + c_{11}), ..., d_n(b_{nn} + c_{nn}))$. Hence all d_i $(1 \leq i \leq n)$ are units and so is D. The converse follows from the previous proposition. □

To provide an example of a diagonal 1-Sylvester matrix over a noncommutative ring that is not invertible, it suffices to choose all the diagonal entries to be one-sided (but not two-sided) invertible elements in any ring that is not Dedekind finite (i.e., there exist elements such that $ab = 1$ but $ba \neq 1$). Such examples exist also over Dedekind finite rings. As mentioned in the Introduction, if $R = \mathbb{M}_2(k)$ for a field k, then E_{12} is 1-Sylvester but not unit. Hence any diagonal matrix (over R) having E_{12} entries on the diagonal is 1-Sylvester but not invertible.

We proceed with a general result that yields several important, albeit mostly negative, consequences.

Proposition 2.3. *Let $1 \leq i \leq n$ and let R be an arbitrary (not necessarily commutative) ring. Any matrix $A \in \mathbb{M}_n(R)$ with only zeros on its i-th row and i-th column is not 1-Sylvester.*

Proof. If A has the i-th row zero, so is AB for every $n \times n$ matrix B. Moreover, if A has the i-th column zero, so is CA for every $n \times n$ matrix C. Hence, for every B, C, the sum $AB + CA$ has the (diagonal) (i, i) entry equal to zero, so the sum is $\neq I_n$, whence A is not 1-Sylvester. □

Corollary 2.4. *The diagonal $n \times n$ matrices with at least one zero diagonal entry are not 1-Sylvester.*

Furthermore, we establish several results concerning nonzero nilpotent matrices that are 1-Sylvester.

Lemma 2.5. *In $\mathbb{M}_2(R)$ over any ring R, the nilpotents E_{12} and E_{21} are mutually anticommutator inverses. As such, these are 1-Sylvester.*

Proof. Just note that $E_{12}E_{21} + E_{21}E_{12} = I_2$. □

Next, we provide an example of a matrix that is 1-Sylvester but not an ACI.

Lemma 2.6. *Over any ring, the nilpotent* $T_3 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *has no anticommutator inverse. However, it is 1-Sylvester.*

Proof. Suppose T_3 is ACI and let $A = [a_{ij}]$ for $1 \leq i, j \leq 3$. Then
$$AT_3 + T_3A = \begin{bmatrix} a_{21} + a_{31} & a_{11} + a_{22} + a_{32} & a_{11} + a_{12} + a_{23} + a_{33} \\ a_{31} & a_{21} + a_{32} & a_{21} + a_{22} + a_{33} \\ 0 & a_{31} & a_{31} + a_{32} \end{bmatrix}.$$
This sum is I_3 only if $a_{31} = 0$. This successively requires $a_{21} = 1$ and then $a_{32} = 0$. Hence the $(3, 3)$ is zero, a contradiction.

However, we can find A, B such that $AT_3 + T_3B = I_3$. For example, for $A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = E_{21}$, the sum $AT_3 + T_3B = (E_{22} + E_{33}) + E_{11} = I_3$. □

Remark. The pair (A, B) given as example in the previous proof is far from being unique. One can replace the third column of A and the first row of B by arbitrary entries and the result of this computation remains unchanged.

In the $n \times n$ case, we can generalize the nonzero nilpotent 1-Sylvester matrix described in the previous lemma.

Theorem 2.7. *Let n be a positive integer and let T_n be the strictly upper triangular $n \times n$ matrix which has all entries above the diagonal equal to 1. Then T_n is 1-Sylvester.*

Proof. Take $A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ (excepting two zeros, the diagonal entries are equal to -1, and the subdiagonal entries are equal to 1) and $B = E_{21}$. Then $AT_n + T_nB = (E_{22} + E_{33} + ... + E_{nn}) + E_{11} = I_n$.

The details of the computation follow. We actually have

$A = \sum_{i=2}^{n-1}(E_{i,i-1} - E_{ii}) + E_{n,n-1} =$

$E_{21} - E_{22} + E_{32} - E_{33} + ... + E_{n-1,n-2} - E_{n-1,n-1} + E_{n,n-1}$ and

$T_n = \sum_{i,j=1,i<j}^{n} E_{ij} = (E_{12} + E_{13} + ... + E_{1n}) + (E_{23} + E_{24} + ... + E_{2n}) + ... + E_{n-1,n}$.

For AT_n, the product starts with

$E_{21}(E_{12} + E_{13} + ... + E_{1n}) - E_{22}(E_{23} + E_{24} + ... + E_{2n}) =$

$E_{22} + E_{23} + ... + E_{2n} - E_{23} - E_{24} - ... - E_{2n} = E_{22}$, and so on. □

Further, recall that every nilpotent matrix over a field is similar to a *block diagonal* matrix $\begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$, where each block B_i is a shift matrix (possibly of different sizes), a special case of the *Jordan canonical form* for matrices. A *shift* matrix has 1's along the superdiagonal and 0's everywhere else, i.e. $S = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$, as an $n \times n$ matrix. When $n = 1$, $S = 0$.

In [2] Theorem 3.3, the following result was proved.

Theorem 2.8. *The following are equivalent for a ring R.*

(1) Every nilpotent matrix over R is similar to a block diagonal matrix with each block a shift matrix (possibly of different sizes).

(2) R is a division ring.

Thus, according to Proposition 2.1, (vi), it follows

Theorem 2.9. *Over any division ring, each nonzero nilpotent matrix is 1-Sylvester.*

Proof. Any block diagonal matrix with each block a shift matrix is strictly upper triangular with only superdiagonal nonzero entries, which are equal to 1, that is, $S = E_{12}+E_{23}+...+E_{n-1,n}$. Then if $T = E_{21}+E_{32}+...+E_{n,n-1}$ is the subdiagonal, we have $TS + SE_{21} = (E_{22} + E_{33} + ... + E_{nn}) + E_{11} = I_n$, as desired. □

It follows from the previous theorem that, over any ring, all shift matrices are 1-Sylvester. However, only shift matrices of even size admit an anticommutator inverse (ACI).

Proposition 2.10. *Over any ring, the shift matrices are ACI iff they are of even size.*

Proof. Let $T = [t_{ij}]$ be an arbitrary $n \times n$ matrix and let S be the shift matrix of size n. We focus on the diagonal entries of the sum $ST + TS$.

These are t_{21}, $t_{32} + t_{21}$,..., $t_{n,n-1} + t_{n-1,n-2}$, $t_{n,n-1}$. If $ST + TS = I_n$ then all these entries equal 1. Hence $t_{21} = 1$, $t_{32} = 0$, $t_{43} = 1$ and so on. If n is odd then $t_{n,n-1} = 0$, a contradiction. If n is even, the alternation ends with $t_{n-1,n-2} = 0$ and $t_{n,n-1} = 1$. All the other entries of T can be chosen equal to zero and so T is an anticommutator inverse for S.

More precisely, in the even case, for $S = E_{12} + E_{23} + ... + E_{2n-1,2n}$, the matrix $T = E_{21} + E_{43} + ... + E_{2n,2n-1}$ (i.e., on the subdiagonal we alternate 1, 0, 1, 0 , ...) is an anticommutator inverse for S. Indeed,

$$ST + TS = (E_{11} + E_{33} + ... + E_{2n-1,2n-1}) + (E_{22} + E_{44} + ... + E_{2n,2n}) = I_{2n}.$$

□

3. The 1-Sylvester 2 × 2 matrices.

In order to describe the 1-Sylvester 2×2 matrices over commutative rings, we start with $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, $C = \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix}$.

The Sylvester equation $AB + CA = I_2$ reduces to a nonhomogeneous linear system of 4 equations and 8 unknowns:

$$\begin{cases} ax_1 + bx_3 + ay_1 + cy_2 & = & 1 \\ ax_2 + bx_4 + by_1 + dy_2 & = & 0 \\ cx_1 + dx_3 + ay_3 + cy_4 & = & 0 \\ cx_2 + dx_4 + by_3 + dy_4 & = & 1 \end{cases}$$

with the system matrix $\begin{bmatrix} a & 0 & b & 0 & a & c & 0 & 0 \\ 0 & a & 0 & b & b & d & 0 & 0 \\ c & 0 & d & 0 & 0 & 0 & a & c \\ 0 & c & 0 & d & 0 & 0 & b & d \end{bmatrix}$, augmented by the column $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

By applying the Kronecker product and vectorization, we will show at the end of this section that the Sylvester equation can be reduced to a matrix equation of the form $PX = Q$.

Next, we describe the *upper triangular* 1-Sylvester 2×2 matrices over commutative domains.

Proposition 3.1. *Let R be a commutative domain. The matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(R)$ is 1-Sylvester iff both of the following conditions hold:*
1. a and b are coprime.
2. There exists $u \in U(R)$ such that $d = au$ and $a \mid 1 + u^{-1}$.

Proof. ($\Rightarrow$) For $c = 0$ the above system becomes $\begin{cases} a(x_1 + y_1) + bx_3 & = & 1 \\ ax_2 + bx_4 + by_1 + dy_2 & = & 0 \\ dx_3 + ay_3 & = & 0 \\ d(x_4 + y_4) + by_3 & = & 1 \end{cases}$.

From the first and fourth equations follows that a, b are coprime (and b, d are coprime).

Multiplying first equation by d and replacing the third equation shows that $a \mid d$. Analogously, multiplying the fourth equation by a and replacing the third equation shows $d \mid a$.

Therefore, in general, a and d are associates (i.e., $d = au$ for some unit u). If $a = 0$, then $d = 0$ and we can take $u = -1$. If $a \neq 0$, then $d \neq 0$. Then from the third equation, $y_3 = -ux_3$ and the forth equation becomes $a(x_4 + y_4) - bx_3 = u^{-1}$. Adding the first equation gives $a(x_1 + y_1 + x_4 + y_4) = 1 + u^{-1}$.

($\Leftarrow$) From conditions 1 and 2 there exist $x_1, x_3 \in R$ and $u \in U(R)$ such that $ax_1 + bx_3 = 1$, $d = au$, and $a \mid 1 + u^{-1}$. Take $x_2 = x_4 = y_1 = y_2 = 0$, $y_3 = -ux_3$, and $y_4 = v - x_1$, where $av = 1 + u^{-1}$. Then, one may check that these choices produces a solution to the required system of equations. Equivalently, one may compute that $\begin{bmatrix} a & b \\ 0 & au \end{bmatrix} \begin{bmatrix} x_1 & 0 \\ x_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -ux_3 & v - x_1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & au \end{bmatrix} = I_2$. □

In particular, as $U(\mathbb{Z}) = \{\pm 1\}$, we characterize the upper triangular integral 2×2 matrices that are 1-Sylvester.

Proposition 3.2. *The upper triangular 1-Sylvester integral 2×2 matrices are:*
(i) $\pm E_{12}$;

(ii) $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ *with coprime a, b and $a \in \{\pm 1, \pm 2\}$;*

(iii) $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ *with coprime a and b.*

Proof. Just note that in case (ii) $u = 1$ and so $a \mid 2$, and in case (iii), $u = -1$. □

Building on Proposition 3.1, we can readily characterize all upper triangular 2×2 ACI matrices over a commutative domain.

Proposition 3.3. *Let R be a commutative domain. The matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathbb{M}_2(R)$ is ACI iff all three of the following conditions hold:*

1. *$2a$ and b are coprime.*
2. *There exists $u \in U(R)$ such that $d = au$ and $a \mid 1 + u^{-1}$.*
3. *Either $u = -1$, or $2a$ is a unit.*

Proof. ($\Rightarrow$) Constructing the system of equations as in Proposition 3.1 proves condition 1. The second condition holds because any ACI matrix is 1-Sylvester. For condition 3, use the equation $(a+d)x_3 = 0$. Clearly, $a + d = 0$ implies that $u = -1$. When $a + d \neq 0$, we have $x_3 = 0$. Then the equation $2ax_2 + bx_3 = 1$ shows that $2a$ is a unit.

($\Leftarrow$) If all three conditions hold, let $x_1, x_3 \in R$ be such that $2ax_1 + bx_3 = 1$. Consider two cases.

When $u = -1$, then $\begin{bmatrix} x_1 & 0 \\ x_3 & -x_1 \end{bmatrix}$ is an anticommutator inverse for $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$.

When $2a \in U(R)$, take $x_1 = 2a - 1$, $x_4 = 2d - 1$, $x_2 = -2bx_1x_4$ and $x_3 = 0$. These choices satisfy the system of equations for x_1, x_2, x_3, and x_4.

Alternatively, one can verify that

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} 2a-1 & -2bx_1x_4 \\ 0 & 2d-1 \end{bmatrix} + \begin{bmatrix} 2a-1 & -2bx_1x_4 \\ 0 & 2d-1 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = I_2.$$

□

Finally, for integral anticommutator inverses we have the following characterization.

Proposition 3.4. *An upper triangular 2×2 matrix $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ has an anticommutator inverse over $\mathbb{Z}$ iff $A = \pm E_{12}$ or else $d = -a$ and $2a$, b are coprime.*

Remarks. 1) Since $b \neq 0$, it follows from the first equation of the linear system above that, over $\mathbb{Z}$, A has an anticommutator inverse only if b is odd. If so, $\gcd(2a, b) = 1$ iff $\gcd(a, b) = 1$.

As an example, $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ is 1-Sylvester by Proposition 3.2 (or directly, since it is invertible) but not ACI (actually, over any ring where 2 is not a unit). Indeed, for any matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, the sum $AB + BA = 2\begin{bmatrix} x+z & x+w \\ 0 & z-w \end{bmatrix} \neq I_2$. Here $\gcd(1, 2) = 1 \neq 2 = \gcd(2, 2)$. This is also an example of *unit that has no anticommutator inverse.*

2) Since in the anticommutator inverse of the proof of the previous proposition, the entry x_2 is arbitrary, the matrices $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ (with coprime $2a$ and b) have infinitely many anticommutator inverses over $\mathbb{Z}$.

Not only invertible matrices may have an anticommutator inverse. We also have the following result.

Proposition 3.5. *Let a be an element of an arbitrary ring R. All upper triangular matrices $A_a = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in \mathbb{M}_2(R)$ are ACI. As such, these are 1-Sylvester.*

Proof. This follows as $\begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} E_{21} + E_{21} \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} = I_2$. It is easy to see that, for a commutative ring R and arbitrary $x, y \in R$, $\begin{bmatrix} x & y \\ 1-2ax & -x \end{bmatrix}$ are all the anticommutator inverses of A_a. □

Remarks. 1) Observe that $\begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$ is a unit iff a is a unit. So all the A_a, where a is not a unit, are not invertible upper triangular matrices that have many anticommutator inverses.

2) An anticommutator inverse of the transpose $(A_a)^T$ is $E_{12} = (E_{21})^T$.

To conclude this section, we establish a result concerning the uniqueness of the anticommutator inverse.

Lemma 3.6. *If $2a$ is a unit then $\frac{a^{-1}}{2}$ is an anticommutator inverse for a.*

Proof. As 2 and a commute, the hypothesis is equivalent to $2, a \in U(R)$. □

In the remainder of this section, we apply two successive simplifications to establish a uniqueness result for ACI matrices. These simplifications also allow to investigate properties of 1-Sylvester 3×3 matrices as well.

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the *Kronecker product* $A \otimes B$ is the $pm \times qn$ block matrix:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Especially in linear algebra and matrix theory, the *vectorization* of a matrix is a linear transformation which converts the matrix into a vector. Specifically, the vectorization of a $m \times n$ matrix A, denoted $vec(A)$, is the $mn \times 1$ column vector obtained by stacking the columns of the matrix A on top of one another:

$$vec(A) = [a_{11}, ...a_{m1}, a_{12}, ..., a_{m2}, ..., a_{mn}]^T.$$

Using the Kronecker product notation and the vectorization operator vec, we can rewrite Sylvester's equation (i.e., $AX + XB = C$) in the form

$$(I_m \otimes A + B^T \otimes I_n) vecX = vecC,$$

where (in general) A is of dimension $n \times n$, B is of dimension $m \times m$, X is of dimension $n \times m$ and I_k is the $k \times k$ identity matrix. In this form, the equation can be seen as a linear system of dimension $mn \times mn$.

If we take matrices of the same size and $C = I_n$, we obtain the 1-Sylvester $n \times n$ matrices studied in this paper. In particular, if we also take $A = B$ we get the ACI matrices. That is, the ACI equation $AB + BA = I_n$ is represented as $(I_n \otimes A + A^T \otimes I_n)vec(B) = vec(I_n)$.

From this representation it follows that A has a unique anticommutator inverse iff $I_n \otimes A + A^T \otimes I_n$ is invertible.

Theorem 3.7. *Let R be a commutative ring and let $A \in \mathbb{M}_2(R)$. Then, A has a unique anticommutator inverse if and only if 2, $Tr(A)$, and $\det(A)$ are units of R. In this case, the unique anticommutator inverse is $\frac{1}{2}A^{-1}$.*

Proof. Written as 2×2 blocks we have

$$I_2 \otimes A + A^T \otimes I_2 = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} + \begin{bmatrix} aI_2 & cI_2 \\ bI_2 & dI_2 \end{bmatrix} = \begin{bmatrix} A + aI_2 & cI_2 \\ bI_2 & A + dI_2 \end{bmatrix}.$$

To check when this matrix is invertible, we have to compute the determinant of the 4×4 matrix

$$M = \begin{bmatrix} 2a & b & c & 0 \\ c & a+d & 0 & c \\ b & 0 & a+d & b \\ 0 & b & c & 2d \end{bmatrix}.$$

Subtracting $col_4(M)$ from $col_1(M)$ and $\mathrm{row}_4(M)$ from $\mathrm{row}_1(M)$, simplify a lot the computation of $\det(M) = 4Tr^2(A)\det(A)$.

By computation we also get $\Delta_x = 2dTr^2(A)$, $\Delta_y = -2bTr^2(A)$, $\Delta_z = -2cTr^2(A)$, $\Delta_w = 2aTr^2(A)$.

Hence, if $2\det(A)$ is a unit (and $Tr^2(A) \neq 0$), we get $x = \frac{d}{2\det(A)}$, $y = -\frac{b}{2\det(A)}$, $z = -\frac{c}{2\det(A)}$, $w = \frac{a}{2\det(A)}$. This gives $B = \frac{1}{2\det(A)}adj(A)$ where the adjugate matrix is $adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Finally, $B = \frac{1}{2}A^{-1}$.

Conversely, if $4Tr^2(A)\det(A)$ is a unit, then 2, $Tr(A($ and $\det(A)$ must be units. □

Remark. The existence of $\frac{1}{2}A^{-1}$ in $\mathbb{M}_2(R)$ does not imply that A has a unique anticommutator inverse. For an explicit example, let R be any ring for which $2 \in U(R)$, and take $A = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$. Then, both $\frac{1}{2}A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$ and $B = -E_{21}$ are anticommutator inverses of A.

In closing, to determine suitable conditions characterizing ACI 3×3 matrices, one may apply Jameson's approach (see [3]) for solving the Sylvester equation. However, the resulting conditions are rather unwieldy. A sample is given below.

Theorem 3.8. *Let R be any commutative ring and A a 3×3 matrix over R. The matrix A is ACI iff 2, $\det(A)$ and $\det(Tr(A)A^2 + \det(A)I_3)$ are units in R.*

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