SOME PROPERTIES PRESERVED BY WEAK NEARNESS

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Abstract: We show that the properties of being injective, surjective, continuous, strongly surjective, stably solvable, carry over to weakly-near maps. We also prove a data dependence theorem for equations of the form $A(x) = z$, where A is weakly-near to some bijective map.

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1 Introduction

Let X be a nonempty set and Z be a Banach space. Let $A, B : X \to Z$ be two operators. S. Campanato [4] introduced the following notion of nearness between operators in order to use it in the study of fully nonlinear elliptic equations [4, 5, 13, 3, 2].

Definition 1.1 (Campanato [4]) *We say that A is near B if there exists $\alpha > 0$ and $0 \leq c < 1$ such that*

$$\|Bx - By - \alpha(Ax - Ay)\| \leq c\|Bx - By\| \tag{1.1}$$

for all $x, y \in X$.

In a joint paper with A. Domokos [3] we generalized this notion using an accretivity-type condition, instead of a contraction-type one.
Let us denote by Φ the set of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that $\varphi(0) = 0$, $\varphi(r) > 0$ for $r > 0$, $\liminf_{r\to\infty} \varphi(r) > 0$ and $\liminf_{r\to r_0} \varphi(r) = 0$ implies $r_0 = 0$. In this paper we shall refer only to the functions φ in Φ.
We say that A *is φ-accretive w.r.t.* B (see [3]), if for every $x, y \in X$ there exists $j(Bx - By) \in J(Bx - By)$ such that

$$\langle Ax - Ay, j(Bx - By)\rangle \geq \varphi(\|Bx - By\|)\|Bx - By\|, \tag{1.2}$$

where $J : Z \rightsquigarrow Z^*$ is the normalized duality map of Z.
The map A *is continuous w.r.t.* B if $A \circ B^{-1} : B(X) \rightsquigarrow Z$ has a continuous selection.
The next definition introduce the weak-nearness notion.

Definition 1.2 *We say that A is weakly-near B if A is φ-accretive w.r.t. B and continuous w.r.t. B.*

This notion extends the property of the differential operator to be "near" (or to "approximate") the map, as well as other approximation notions used in nonsmooth theory of inverse or implicit functions (for details in this direction, see [2, 3, 6, 7]).

A. Tarsia proved in [12] that injectivity, surjectivity and open image carry over to near maps. We proved in [1] that the property of being a strong surjection or stably solvable is preserved by nearness. The notions of strong surjection and stably solvable map were introduced by M. Furi, M. Martelli and A. Vignoli in [8] in order to define the spectrum for a nonlinear operator. Also, these concepts are related to that of zero-epi map, which is due to the same authors [9] and is very important in the study of solvability of nonlinear equations.
Let X be a normed space. A continuous map $B : X \to Z$ is called a *strong surjection* if the equation $B(x) = g(x)$ has a solution for any continuous map $g : X \to Z$.
A continuous map $B : X \to Z$ is said to be *stably solvable* if the equation $B(x) = g(x)$ has a solution for any completely continuous map $g : X \to Z$ with quasinorm $|g| = 0$.
Recall that the *quasinorm* of a map g is defined by

$$|g| = \lim sup_{||x|| \to \infty} \frac{||g(x)||}{||x||}.$$

In this paper we shall prove that nothing is lost if the operators are weakly-near, instead of near. Injectivity, surjectivity, strong surjectivity and some other qualities, are also preserved by weak-nearness. We shall also prove a data dependence theorem for equations of the form $A(x) = z$, where A is weakly-near to some bijective map.

2 Main results

We shall start with some useful lemmas.

Lemma 2.1 *if A is near B, then B is injective w.r.t. A.*

Lemma 2.2 *If A is weakly near B, then any selection of $A \circ B^{-1} : B(X) \rightsquigarrow Z$ is φ-acretive.*

Proof. Let $f : B(X) \to Z$ be a selection of $A \circ B^{-1}$. In order to prove that f is φ-acretive, let us consider $u, v \in B(X)$. We have $f(u) \in A \circ B^{-1}(u)$ and $f(v) \in A \circ B^{-1}(v)$, i.e. there exist $x \in B^{-1}(u)$ and $y \in B^{-1}(v)$ such that $f(u) = Ax$, $u = Bx$, $f(v) = Ay$ and $v = By$. Then

$$\langle Ax - Ay,\ j(Bx - By)\rangle \geq \varphi(\|Bx - By\|)\|Bx - By\|$$

implies

$$\langle f(u) - f(v),\ j(u - v)\rangle \geq \varphi(\|u - v\|)\|u - v\|. \tag{2.3}$$

□

Lemma 2.3 *Let B be surjective and let us assume that B is injective w.r.t. A, and A is weakly near B. Then $A \circ B^{-1} : Z \to Z$ is an homeomorphism.*

Proof. The hypothesis assure that $A \circ B^{-1}$ is single-valued, φ-acretive and continuous. Using Theorem 3 [11], it is also surjective. Relation (2.3) implies easily the injectivity of $f = A \circ B^{-1}$. Also, from (2.3) we obtain

$$\|f(x) - f(y)\| \geq \varphi\left(\|x - y\|\right).$$

Thus

$$\varphi\left(\|f^{-1}(u) - f^{-1}(v)\|\right) \leq \|u - v\|,$$

which implies that $f^{-1} = (A \circ B^{-1})^{-1}$ is continuous. □

Remark. Without the hypotheses that B is injective w.r.t. A, conclusion becomes: there exists a selection of $A \circ B^{-1}$ which is an homeomorphism.

The next propositions and theorems are our main results.

Proposition 2.1 *Let A be weakly-near to B. If B is injective, then A is injective.*

Proof. Let $x, y \in X$ with $Ax = Ay$ and put into (1.2). We obtain that $Bx = By$, what implies that $x = y$. □

Proposition 2.2 *Let A be weakly-near to B. If B is surjective, then A is surjective.*

Proof. Let $f : Z \to Z$ be a continuous selection of $A \circ B^{-1}$ (the existence of such selection is assured by the fact that A is continuous w.r.t. B). Theorem 3 in [11] says that any continuous and φ-accretive map is surjective, which is the case of our f. Then $A \circ B^{-1}$ is surjective, what implies that A is surjective. □

Proposition 2.3 *Let A be weakly-near to B and B be injective w.r.t. A. If $B(X)$ is open, then $A(X)$ is open.*

Proof. In this case $A \circ B^{-1}$ is single-valued, so $f = A \circ B^{-1}$. Moreover, $f(B(X)) = A(X)$. The fact that $B(X)$ is open assures that $f(B(X))$ is open, by applying Corollary 3 from [11]. □

Proposition 2.4 *Let A be weakly-near B and B be injective w.r.t. A. Let X be a topological space. If B is continuous, then A is continuous.*

Proof. Because A is weakly near B, we have that $A \circ B^{-1}$ is continuous (also, our hypothesis assures that is single-valued). Using the continuity of B and relation $A = A \circ B^{-1} \circ B$, we obtain that A is continuous. □

Proposition 2.5 *Let B be bijective, X be a topological space and A be weakly-near to B. If B^{-1} is continuous, then A^{-1} is continuous.*

Proof. Lemma 2.3 assures that $A \circ B^{-1}$ is an homeomorphism. Thus $(A \circ B^{-1})^{-1} = B \circ A^{-1}$ is continuous. Then $A^{-1} = B^{-1} \circ B \circ A^{-1}$ is continuous. □

Proposition 2.6 *Let X be a topological space and A be weakly-near to B. If B is an homeomorphism, then A is an homeomorphism.*

Theorem 2.1 *Let X be a normed space, A be weakly-near B and B be injective w.r.t. A. If B is a strong surjection, then A is a strong surjection.*

Proof. We have to prove that A is continuous and A has at least one coincidence point with every $g : X \to Z$ continuous and compact. Using Proposition 2.4, the continuity of B assures that A is continuous.
Let $g : X \to Z$ be continuous and compact.
Let us denote $f = A \circ B^{-1} : Z \to Z$, which is an homeomorphism (Lemma 2.3). Then $f^{-1} \circ g : X \to Z$ is continuous and compact. Thus, it has a coincidence point, $x^* \in X$ with the strong surjection B, i.e.

$$(f^{-1} \circ g)(x^*) = B(x^*).$$

Then $g(x^*) = f(B(x^*))$, which means $g(x^*) = A \circ B^{-1}(B(x^*))$, thus

$$g(x^*) = A(x^*).$$

g is arbitrar, we have that A is a strong surjection. □

Theorem 2.2 *Let X be a normed space, A be weakly-near B and B be injective w.r.t. A. If B is stably solvable, then A is stably solvable.*

Proof. Let $g : X \to Z$ be completely continuous with quasinorm $|g| = 0$. The arguments follow like in the previous theorem, noticing that $f^{-1} \circ g$ is completely continuous with quasinorm $|f^{-1} \circ g|$ equals with 0. □

Let $z \in Z$ and $A_1, A_2 : X \to Z$. Let us consider the equation

$$A_1(x) = z,$$

whose solvability is assured by the weak-nearness between the operator A_1 and a bijective operator $B : X \to Z$. Let x_1^* be a solution of this equation. Let us consider, also, the equation

$$A_2(x) = z,$$

which is assumed to be solvable. Let x_2^* be a solution. In the following theorem we shall give an estimation of "the distance" between x_1^* and x_2^*. This distance depends of the operator B.

Theorem 2.3 *Let us assume that the following conditions are fullfiled.*
(i) B is bijective;
(ii) A_1 is weakly-near to B with $\varphi(t) = \alpha t,\ 0 < \alpha < 1$;
(iii) equation $A_2(x) = z$ has at least a solution;
(iv) there exists $\eta > 0$ such that $||A_1(x) - A_2(x)|| \le \eta$ for all $x \in X$.
Then we have the estimation

$$||B(x_1^*) - B(x_2^*)|| \le \frac{1}{\alpha}\eta.$$

Proof. From (1.2) we obtain easyly the relation

$$\|B(x) - B(y)\| \le \frac{1}{\alpha}\|A_1(x) - A_1(y)\|,$$

for all $x, y \in X$. If we write again this relation for x_1^* and x_2^* and use that $A_1(x_1^*) = z = A_2(x_2^*)$ and the hypotheses (iv), we obtain the estimation. □

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