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Lemniscate Starlike Functions: The Proof of Sokół's Conjecture

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Abstract

Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{S}_L^*$ be the class of functions satisfying $zf'(z)/f(z) \prec \sqrt{1+z}$, introduced by Sokół and Stankiewicz. In 2009, Sokół conjectured that $|a_n| \leq 1/(2(n-1))$ for all $n \geq 2$; the cases $n = 2, 3, 4$ were proved by Sokół, and $n = 5, 6$ later required much more involved optimization arguments. We prove the conjecture for every $n \geq 2$ by a single argument independent of the coefficient order. The proof rests on two exact factorizations of the defining subordination in terms of Schwarz functions, one of them via the Cayley transform of $zf'(z)/f(z)$, and on a truncated H^2-contraction estimate based on Parseval's identity. This yields the sharp inequality

$$4(n-1)^2|a_n|^2 + 3\sum_{k=1}^{n-2} k^2|a_{k+1}|^2 \leq 1, \qquad n \geq 2,$$

and hence the conjectured bound, as well as $\sum_{n=2}^{\infty}(n-1)^2|a_n|^2 \leq 1/3$. For each $N \geq 2$, equality at $n = N$ holds if and only if f is a rotation of $f_N(z) = z\exp\int_0^z \big(\sqrt{1+t^{N-1}} - 1\big)/t\,dt$.

References

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