Workshop dedicated to the memory of Professor Gabriela Kohr
(4th edition)
Geometric Function Theory in Several Complex Variables and Complex Banach Spaces

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Q-difference operator extension of Nehari's inequality and applications to majorization results

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Abstract

We gave an extension of the well-known Nehari's inequality [3, p. 168] using the D_q Jackson's *q-derivative operator* and this result will be used to the prove some majorisation problems:

Lemma 1. *If ω is an analytic function in $\mathbb{D}$, such that $|\omega(z)| < 1$, $z \in \mathbb{D}$, then*

$$\left| D_q(\omega(z)) \right| \leq \frac{\left|1 - \overline{\omega(zq)}\omega(z)\right|}{1 - |z|^2 q},\ z \in \mathbb{D},\ (0 < q < 1).$$

For $q \to 1^-$ in the above result reduces to the Nehari's inequality.

Let $\mathcal{P}$ be the subclass of all analytic functions χ in the open unit disk $\mathbb{D}$, such that χ has positive real part in $\mathbb{D}$ with $\chi(0) = 1$, and let $\mathcal{A}$ denotes the class of functions f analytic in $\mathbb{D}$ usually normalized by $f(0) = f'(0) - 1 = 0$. For a given $\chi \in \mathcal{P}$ and $q \in (0, 1)$ we define the family $\mathcal{S}_q(\chi) \subset \mathcal{A}$ by

$$\mathcal{S}_q(\chi) := \left\{ k \in \mathcal{A} : \frac{z D_q k(z)}{k(z)} \prec \chi(z) \right\}.$$

Assuming that k and h are two analytic functions in $\mathbb{D}$, then k is said to be *majorized* by h in $\mathbb{D}$, denoted by $k(z) \ll h(z)$, if there exists an analytic function μ in $\mathbb{D}$ such that $|\mu(z)| \leq 1$ and $k(z) = \mu(z)h(z)$ for all $z \in \mathbb{D}$ (see [1]).

A consequence of the above lemma is the following modified version of majorization problem for the class $\mathcal{S}_q(\chi)$ connected with Theorem 1.1 of [2]:

Theorem 1. *Let l be analytic in $\mathbb{D}$ with $l \not\equiv 0$, and let $h \in \mathcal{S}_q(\chi)$. If $l(z) \ll h(z)$ in $\mathbb{D}$ such that $l \not\equiv c\,h$ with $|c| = 1$, and $q \in (0, 1)$, then*

$$|D_q l(z)| \leq |D_q h(z)|,\ |z| \leq r \leq r^*,$$

where r^* is the positive root of the equation

$$(1 - \eta)\rho q r^2 + (1 + \eta)r - (1 - \eta)\rho = 0,$$

with $\eta = \eta(r, q) := \max\limits_{|\zeta| = qr} |\mu(\zeta)|$ and $\rho = \rho(r) := \min\limits_{|z| = r} |\chi(z)|$. The function μ is those that realize the majorization $l(z) \ll h(z)$ in $\mathbb{D}$, shown in the above definition.

The theorem is followed by many particular and special cases obtained for different choice of the parameters and the involved functions.

1. T.H. MacGregor, *Majorization by univalent functions*, Duke Math. J., **34**(1)(1967), 95–102.
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2. N. Hameed Mohammed and E.A. Adegani, *Majorization problems for class of q-starlike functions*, Afr. Mat., **34**(2023), Art. ID 66, 7 pages.
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3. Z. Nehari, Conformal mapping, MacGraw-Hill Book Company, New York, Toronto and London, 1952.