On Neumann's Series and Double Layer Potentials

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1.) The Laplacian: Dirichlet problem:

$\Delta u = 0$ in $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3 ; $u|_\Gamma = \varphi$ on $\Gamma = \partial\Omega$

C.F. Gauss 1837:

$$u(x) = -\frac{2^{1-n}}{\pi} \int\limits_\Gamma \mu(y) \partial_{n_y} E(x,y) do(y)$$

$$= -\frac{2^{1-n}}{\pi} \int\limits_{y \in \Gamma} \mu d\Omega_x \, , \; d\Omega_x = \tfrac{n(y)\cdot(y-x)}{|y-x|^n} do(y) \, , \; x \in \Omega$$

$$E(x,y) = \frac{2^{1-n}}{\pi} \begin{cases} -\ln|x-y| & \text{for } n=2, \\ |x-y|^{-1} & \text{for } n=3. \end{cases}$$

$$\mu_{k+1} = \Big(\tfrac{1}{2} I + K\Big)\mu_k + \varphi \qquad \text{on } \Gamma \, , \quad \mu \in \mathfrak{X}(\Gamma)$$

$$L\mu(x) = \tfrac{1}{2}\mu(x) + K\mu(x) = -\frac{2^{1-n}}{\pi} \int\limits_{y \in \Gamma \setminus \{x\}} \big(\mu(y) - \mu(x)\big) d\Omega_x(y)$$

$$K\mu(x) = -\frac{2^{1-n}}{\pi} \int\limits_{\Gamma \setminus \{x\}} \mu d\Omega_x + \Big(\tfrac{2^{1-n}}{\pi}\Omega_x - \tfrac{1}{2}\Big)\mu(x) \, , \; \Omega_x = \int\limits_{\Gamma \setminus \{x\}} d\Omega_x$$

Neumann problem: $\partial_n u|_\Gamma = d\Psi \; (= \psi do)$ on Γ , $\int\limits_\Gamma d\Psi = 0$.

Boundary Flux: $\langle v, \Psi\rangle = \int\limits_\Gamma v d\Psi = \int\limits_\Gamma v \partial_n u$ Plemelj 1911, Radon 1919:

$$u(x) = \int\limits_{y\in\Gamma} E(x,y) d\mathsf{P} \quad \Big(= \int\limits_{y\in\Gamma} E(x,y)\varrho(y) do(y)\Big).$$

$$\mathsf{P}_{k+1} = \Big(\tfrac{1}{2} I - K^*\Big)\mathsf{P}_k + \Psi \qquad \text{on } \Gamma\,,\ \mathsf{P}_{k+1} \in \mathfrak{X}_0^*\,,\ \int\limits_\Gamma d\mathsf{P}_k = 0.$$

- Assumptions on $\Gamma = \partial\Omega$?
- Function space $\mathfrak{X}$ on Γ ?
- Fredholm theory, spectrum, and convergence of Neumann's series?
- Discretization?

$$\lambda v = Av + f \qquad \text{in the Banach space } \mathfrak{X}\,, \ \ \|v\|_{\wp},$$

$$\|A\|_{\wp\mathrm{ess}} := \inf_{C} \|A - C\|_{\wp}\,, \ \ C \quad \text{completely continuous}\,.$$

Fredholm theory for $|\lambda| > \|A\|_{\wp\mathrm{ess}} =$ (Fredholm radius)$^{-1}$,

$$r_{\wp}(A) := \lim_{n\to\infty} (\|A^n\|_{\wp})^{1/n} \qquad \text{spectral radius}\,.$$

$\boldsymbol{\Omega} \subset \mathbb{R}^{\mathbf{2}}$: Γ at least a rectifiable Jordan curve

C. Neumann 1870,1877,1888: $\mu_{k+1} = L\mu_k + \varphi\,,\ k \in \mathbb{N},$

$$\Gamma \quad \text{convex}\ , \mathfrak{X} = C^0(\Gamma)\ ,\ osc(v) := \sup_{x,y\in\Gamma} |v(x) - v(y)|,$$
$$osc\Big(\tfrac{1}{2\pi} \int\limits_{\Gamma\setminus\{x\}} |d\Omega_x|\Big) \le \delta < \tfrac{1}{2},$$

$$\|v\|_{\wp} := osc(u) + \beta \sup |u(x)|\ ,\ \ 0 < \beta < 1 - \delta\ :\ \|L\|_{\wp} \le q < 1\,.$$

J. Plemelj 1911:

$$\Gamma \quad \text{arbitrary}\ \in C^{1+\alpha}\ ,\ \ \mathfrak{X} = C^0(\Gamma)\ ,\ \ \|K\|_{\wp\mathrm{ess}} = 0\ ,\ \ \|v\|_{\wp} = \sup_{x\in\Gamma} |u(x)|$$
$$r_{\wp}(L) = \lim_{n\to\infty} (\|L^n\|_{\wp})^{1/n} < 1\ ,\ L = \tfrac{1}{2}I + K\,,$$
$$\lambda_\ell = \frac{J^+ - J^-}{J^+ + J^-}\ ,\ J^{\pm} := \int\limits_{\Omega^{\pm}} |\nabla u|^2 dx$$

J. Radon 1913,1919: Γ bounded rotation, $\mathfrak{X} = C^0(\Gamma), \mathfrak{X}^* =$ RM,

$$x(s) = x(s_0) + \int_{s_0}^{s} \big(\cos\vartheta(t)\ ,\ \sin\vartheta(t)\big)^\top dt\ ;\ \int_\Gamma |d\vartheta| < \infty,$$

$$\|K\|_{\wp\mathrm{ess}} = \Big|\frac{\alpha}{2\pi}\Big| < \frac{1}{2} \qquad \text{(no cusps)} \quad \Rightarrow \|L\|_{\wp\mathrm{ess}} < 1,$$

Gauss–Green's formula, $(u \in H^1(\Omega))$ and $KV = VK^*$ on $\mathfrak{X}^* =$ RM,

$$\lambda \in \sigma(2K) \wedge |\lambda| > \left|\frac{\alpha}{\pi}\right| \Rightarrow \lambda_\ell \in \mathbb{R}\ ,\ \lambda_0 = -1 \text{ simple}, |\lambda_\ell| = \left|\frac{J^+ - J^-}{J^+ + J^-}\right| < 1\,.$$

J. Radon 1919: $K(\lambda)$, spectral transformation: $\lambda = 2\frac{\eta}{1-\eta}$,

$$r_\wp(L) = \lim_{n\to\infty} (\|L^n\|_\wp)^{1/n} = q < 1.$$

J. Kral 1987: $\lim\limits_{r\to 0} \sup\limits_{x\in\Gamma} \frac{2^{(1-n)}}{\pi} \int\limits_{0<|x-y|\le r} |d\Omega_x(y)| < \frac{1}{2}$,

necessary and sufficient! for admissible $\Gamma \in \mathbb{R}^n$ for $n = 2$ or 3.

Analysis based on

$$Au := (I - Q - C)u = \varphi$$

where $\|Q\| \le q < 1$ and C completely continuous.

Consequences:

- For the Neumann series exist $c > 0$ and $q_0 < 1\,,$
  $u_{j+1} = Lu_j + \varphi \to u$ and
  $\|u - u_j\| \le c q_0^j (\|u\| + \|u_0\|)\,,\ \|u\| = \sup_{x\in\Gamma} |u(x)|$
  convergence in $C^0(\Gamma)$ and $\Re(\Gamma)$, and in RM_0, respectively.
- Boundary element collocations with piecewise constant or piecewise linear elements are stable and convergent in $\Re$ and in RM_0, respectively; $\Re =$ regulated functions $=$ closure w.r.t. to sup norm of $\{$piecewise constant functions$\}$.
  H. Brakhage 1960, G. Bruhn + Wd. 1967, P.M. Anselone 1968 and book 1971, K.E. Atkinson 1967 and book 1997 S. Micula 1997, radiosity 1997–2006, nonlinear Riemann-Hilbert problems, 2008 – now.

$\boxed{\mathbf{\Omega} \subset \mathbb{R}^{\mathbf{3}} :}$ $\quad \mathfrak{X} = C^0(\Gamma) \subset \Re(\Gamma)\ ,\ \mathfrak{X}^* = RM$

C. Neumann 1870, 1877, 1888: $\mathbf{\Gamma}$ **convex** (and Lipschitz),

$$osc\Big(\tfrac{1}{4\pi} \int\limits_{\Gamma\setminus\{x\}} |d\Omega_x|\Big) \leq \delta < \tfrac{1}{2}\ , \qquad \|L\|_{\wp} \leq q < 1$$

I. Netuka 1975:
Fredholm radius of $K = \Big\{ \lim\limits_{\delta\to 0} \sup\limits_{x\in\Gamma} \frac{1}{4\pi} \int\limits_{0<|x-y|<\delta} d\Omega_x(y) \Big\}^{-1}$

Γ Lipschitz or piecewise smooth: V. Maz'ya 1962, 1966; Wd 1965, 1967;
J. Kral 1965

Fredholm radius of K with respect to $\|\cdot\|_{\sup}$ on $C^0(\Gamma)$

$$= \Big\{ \lim_{\delta\to 0} \sup_{x\in\Gamma} \tfrac{1}{4\pi} \int\limits_{0<|x-y|<\delta} |d\Omega_x(y)| \Big\}^{-1}$$

Weighted supremum norms on $\mathfrak{X} = \mathbf{C^0}(\boldsymbol{\Gamma}) \subset \Re(\Gamma)$:

$$r_{\text{ess}}(A) := \underline{\lim_{n\to\infty}}(\|A^n\|_{\wp\text{ess}})^{\frac{1}{n}} = \inf_{\wp} \|A\|_{\wp\text{ess}} \text{ Gohberg+Markus 1960}$$

$$= \inf\{r \geq 0 \text{ where for } \lambda \in \mathbb{C} \text{ with } |\lambda| > r \,:\, (A - \lambda I) \text{ is Fredholm}\}$$

J. Kral + D. Medkova 2000, 2001:

$$\forall \varepsilon > 0 \; \exists \; w \in L^\infty(\Gamma) \wedge 0 < c^- \leq w(x) \leq 1 \; ;$$

$$\|\mu\|_{C_w^0} := \sup_{x\in\Gamma} |\mu(x)w(x)|$$

$$r_{\text{ess}}(L) - \varepsilon \leq \|L\|_{C_w^0\text{ess}} = \lim_{\delta\to 0_+} \Big\{ \sup_{x\in\Gamma\setminus\gamma} \frac{1}{4\pi} \int\limits_{0<|x-y|\leq\delta} \frac{w(x)}{w(y)} |d\Omega_x(y)| \Big\} + \frac{1}{2}$$

$\mathbf{r}_{\mathrm{ess}} < \mathbf{1}$ if Γ piecewise smooth and weighted maximum norm:

convex	C. Neumann 1870/1888,	$w \equiv 1$
Lyapounov $C^{1+\alpha}$	J. Plemelj 1911,	$w \equiv 1$
edges, no corners	T. Carleman 1916,	$w \equiv 1$
edges and corners such that $r_{\mathrm{ess}} < 1$ with		$w \equiv 1$
Yu.D. Burago, V.D. Maz'ya, V.G. Saposhnikova 1962, 1966,		
Yu.D. Burago, V.D. Maz'ya 1969, J. Kral 1966, Wd 1965, 1968		
"smooth" conical points	N. Grachev+V. Maz'ya 1986	
rectangular brigg surfaces	J. Kral et al. 1986,1988	w sectorial const.
polyhedral	A. Rathsfeld 1992,1995	$w =?$
	O. Hansen 2001	w sectorial const.
$\mathbb{R}^3$–diffeomorphic local image of polyhedron Γ	D. Medkova 1990, 1992	$w =?$

Consequences:

For w–compatible triangulations of Γ, collocation with piecewise constants and continuous piecewise linears are stable and convergent; J. Kral+Wd. 1988, J. Elschner 1992, A. Rathsfeld 1995.

Olaf Hansen (2001):

$$\Omega_x(\Delta_j) = \int_{-\frac{\varphi_0}{2}}^{\frac{\varphi}{2}} \frac{|\cos\theta|}{1-\sin\theta\cos(\tau-\varphi)} d\tau$$

$$x = |x|(\sin\theta\cos\varphi\ ,\ \sin\theta\sin\varphi\ ,\ \cos\theta)$$

$$m_{\ell,j} := \frac{1}{4\pi}\begin{cases}\sup\{\Omega_x(\Delta_j)\,|\,x\in\Delta_\ell\} & \text{for } \ell \neq j\,,\\ 0 & \text{otherwise};\end{cases} \quad \mathcal{M} := (\!(m_{\ell,j})\!)$$

Perron–Frobenius: $\exists\ \varrho > 0$ and $\vec{w} = (w_j) > 0\ :\ \mathcal{M}\vec{w} = \varrho\vec{w}$.

> If the spectral radius of $\mathcal{M}$ is less than $\frac{1}{2}$ and if
>
> $$\frac{1}{2}(\Big(1 - \frac{\beta}{\pi}\max\ \Big\{\frac{1}{w_j}\ ,\ \frac{1}{w_{j+1}}\Big\} < \frac{1}{2}\,,$$
>
> take $w(x) := w_j$ for $x \in \Delta_j$. Then $r_{\mathrm{ess}}(L) < 1$.

$\mathfrak{X} = L_2(\Gamma) = \mathfrak{X}^*$ and $\Omega \subset \mathbb{R}^2$:

Shelepov 1969: Γ piecewise smooth, $\|K\|_{L_2 \mathrm{ess}} = \frac{1}{2} \sin \left| \frac{\alpha}{2} \right| < \frac{1}{2}$

$\alpha =$ angular jump of the tangent's direction

Fabes, Sand, Seo 1992: Γ Lipschitz and **convex**:

$r_{L_2}(L) = \lim_{n\to\infty} (\|L^n\|_{L^2})^{1/n} < 1$

Consequences:

- The Neumann series converge in $L_2(\Gamma)$
- Boundary element **Galerkin** methods in $L_2(\Gamma)$ are stable and convergent

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Verchota 1984: Γ Lipschitz. Then $\frac{1}{2}I \pm K$ and $\frac{1}{2}I \pm K^*$ are Fredholm and invertible on $L_2(\Gamma)(\text{ or } L_{2(0)}(\Gamma))$ but no decomposition.

- Only for least squares methods in $L_2(\Gamma)$ stability and convergence is shown.

Maz'ya + Solov'ev 1988, 1997, 1998: Γ has cusps, L_p–theory

$\mathfrak{X} = L_2(\Gamma) = \mathfrak{X}^*$ and $\Omega \subset \mathbb{R}^2$:

For $\Gamma \in C^1$: E.B. Fabes, M. Jodeit, N.M. Riviere 1978,

B.E.J. Dahlberg 1979, M. Mitrea 2014 geometric conditions: K compact

$\Gamma \in C^{1+\alpha}$ or Γ **convex** polyhedron: Fabes, Sand, Seo 1992:

$$r_{L_2}(L) = \lim_{n\to\infty}(\|L^n\|_{L^2})^{1/n} < 1,$$

Γ piecewise smooth Lipschitz and singularities at corners and edges square integrable on Γ : V.G. Maz'ya+B.A. Plamenevskii 1975,1977

Γ Lipschitz with sufficiently small Lipschitz character: I. Mitrea 1999

Consequences:

- Boundary element **Galerkin** methods only in these cases are in $L_2(\Gamma)$ stable and convergent.
- The Neumannn series converge in $L_2(\Gamma)$.

Γ Lipschitz: B.E.J. Dahlberg 1979, G. Verchota 1984:
$\left(\frac{1}{2}I \pm K\right)$ and $\left(\frac{1}{2}I \pm K^*\right)$ are invertible on $L_{2(0)}(\Gamma)$
and on $H^s_{(0)}(\Gamma)$ for $0 \leq s \leq 1$ and $-1 \leq s \leq 0$, respectively.

- Stability and convergence for Galerkin methods in $L_2(\Gamma)$ are not known, only for least squares methods.

Generalizations:

$\underline{C^0(\Gamma)}$ and $\Re(\Gamma)$: More general domains and boundary conditions:
D. Medkova 1999, 2005, 2007

Elasticity and Stokes flows:G.I.Kresin+V.G.Maz'ya 1979+G.I.Kresin 1995

$\underline{L_2(\Gamma)}$: Elasticity and Stokes flows: B.E.J. Dahlberg + C.E. Kenig + G.C. Verchota 1988;

I. Mitrea 1999; T.K. Chang, H.J. Choe 2007,

M. Kohr et al. 2010 – now, L_p^S and Besov spaces, variable viscosity.

Nonsmooth Riemannian manifolds: D.Mitrea,M.Mitrea,M.Taylor 2001

2.) Boundary integral equations of the first and second kind for formally positive elliptic selfadjoint systems of second order with Dirichlet or Neumann conditions on the Lipschitz boundary Γ:

$$\mathcal{L}u(x) := -\sum_{i,j=1}^{n} \partial_j a_{ji}(x)\partial_i u(x) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x)$$

selfadjoint strongly elliptic,

$$T_x u(x) := \sum_{i,j=1}^{n} n_j(x)a_{ji}(x)\partial_i u(x);$$

$$(Vw)(x) := \int_\Gamma E(x,y)w(y)ds_y \qquad \text{for } x \in \Omega^+, \Omega^-, \Gamma,$$

$$(Wv)(x) := \int_\Gamma \big(T_y E(x,y)\big)v(y)ds_y \qquad \text{for } x \in \Omega^+ \cup \Omega^-,$$

$$(Ku)(x) := \text{p.v.}\int_{\Gamma\setminus\{x\}} \big(T_y E(x,y)\big)u(y)ds_y = \lim_{x'\to x} Wu(x') \pm \tfrac{1}{2}u(x)\,,\; x' \in \Omega^\pm,$$

$$(K'w)(x) := \text{p.v.}\int_{\Gamma\setminus\{x\}} \big(T_x E(x,y)\big)w(y)ds_y = \lim_{x'\to x} (W'w)(x') \mp \tfrac{1}{2}w(x)\,,\; x \in \Gamma,$$

$$(W'w)(x) := \int_\Gamma \big(T_x E(x,y)\big)w(y)ds_y \qquad \text{for } x \in \Omega^+ \cup \Omega^-,$$

$$Du(x) := -(T_x \circ Ku)(x) \qquad \text{for } x \in \Omega^+, \Omega^-, \Gamma.$$

.

Mapping properties

M. Costabel 1988:

$$\begin{aligned} V &: H^{-\frac{1}{2}+s}(\Gamma) \to H^{\frac{1}{2}+s}(\Gamma)\,; \\ D = -T_x \circ K &: H^{\frac{1}{2}+s}(\Gamma) \to H^{-\frac{1}{2}+s}(\Gamma)\,; \\ K &: H^{\frac{1}{2}+s}(\Gamma) \to H^{\frac{1}{2}+s}(\Gamma)\,; \\ K' &: H^{-\frac{1}{2}+s}(\Gamma) \to H^{-\frac{1}{2}+s}(\Gamma) \quad \text{for } |s| < \tfrac{1}{2}. \end{aligned}$$

Calderon projection: For u solution of $\mathcal{L}u = 0$ in Ω:

$$\begin{pmatrix} u \\ T_x u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2} + K' \end{pmatrix} \begin{pmatrix} u \\ T_x u \end{pmatrix} \quad \text{on the boundary } \Gamma, \text{then}$$

$$KV = VK'\,,\; DK = K'D\,,\; VD = \tfrac{1}{4}I - K^2\,,\; DV = \tfrac{1}{4}I - K'^2.$$

Coerciveness:

Let $\langle \cdot\,,\,\cdot \rangle := (\cdot\,,\,\cdot)_{L_2(\Gamma)}$ duality

$\langle V\varrho\,,\,\varrho \rangle \geq c_1^V \|\varrho\|^2_{H^{-\frac{1}{2}}(\Gamma)}$ Nedelec–Planchard 1973, Hsiao–Wd. 1976

$\langle D\mu\,,\,\mu \rangle \geq c_1^D \|\mu\|^2_{H^{\frac{1}{2}}(\Gamma)}$ $\forall\, \mu \in H^{1/2}/\gamma\mathfrak{R}$ Costabel 1988, H.Han1988.

The coerciveness properties provide with the Lax–Milgram lemma the existence of

$$\begin{aligned} V^{-1} &: H^{1/2}(\Gamma) &&\to H^{-1/2}(\Gamma) \quad \text{and} \\ D^{-1} &: H^{-1/2}(\Gamma)/\gamma\mathfrak{R} &&\to H^{1/2}/\gamma\mathfrak{R}. \end{aligned}$$

$\mathfrak{R}$ finite dimensional eigenspace of solutions to

$$\mathcal{L}u = 0 \text{ in } \Omega \text{ and } Tu = 0 \text{ on } \Gamma.$$

Aquivalent norms:

$$\begin{aligned} \|u\|^2_{H^{1/2}(\Gamma)} &\simeq \langle V^{-1}u, u\rangle &= \|u\|^2_{V^{-1}}, \\ \|w\|^2_{H^{-1/2}(\Gamma)} &\simeq \langle Vw, w\rangle &= \|w\|^2_{V}. \end{aligned}$$

Mapping properties and coerciveness finally yield with $c_0 = c_1^D c_1^V < \frac{1}{4}$:

$$\begin{aligned} \|(\tfrac{1}{2}I + K)u\|^2_{V^{-1}} &\leq \|(\tfrac{1}{2}I + K)u\|_{V^{-1}}\|u\|_{V^{-1}} - c_0\|u\|^2_{V^{-1}}, \\ a^2 &\leq ab - c_0 b^2 \quad \text{i.e. } (a/b)^2 - a/b + c_0 \leq 0, \end{aligned}$$

hence,

$$\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - c_0} \leq a/b \leq \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_0} =: q < 1.$$

Steinbach+Wd. 2001:

$$
\begin{aligned}
(1-q)\|u\|_{V^{-1}} \le \|(\tfrac{1}{2}+K)u\|_{V^{-1}} &\le q\|u\|_{V^{-1}} && \text{for } u \in H^{1/2}(\Gamma),\\
\|(\tfrac{1}{2}+K')w\|_{V} &\le q\|w\|_{V} && \text{for } w \in H^{-1/2}(\Gamma);\\
\|(\tfrac{1}{2}-K')w\|_{V} &\le q\|w\|_{V} && \text{for } w \in H^{-1/2}(\Gamma)/\gamma\mathfrak{R}\\
(1-q)\|u\|_{V^{-1}} \le \|(\tfrac{1}{2}-K)u\|_{V^{-1}} &\le q\|w\|_{V^{-1}} && \text{for } u \in H^{1/2}(\Gamma)/\gamma\mathfrak{R}.
\end{aligned}
$$

Remark: These results hold as well for strongly elliptic second order systems, e.g. for the system of linear elasticity and the Stokes system.

The Green formula and Poincaré identities together with cerciveness and existence of V^{-1} and D^{-1} on $H^{1/2}(\Gamma)$ and $H_0^{-1/2}(\Gamma) = \{\varphi \in H^{-1/2}(\Gamma) \mid \langle \varphi, \gamma u \rangle = 0 \; \forall \, u \in \mathfrak{R}\}$, respectively, imply further convergent Neumann series:

Theorem (Costabel 2006, Poincaré 1896):

$$(\tfrac{1}{2}I - K)^{-1} = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}I + K)^{\ell} = 2 \sum_{\ell=0}^{\infty} (2K)^{\ell} \quad \text{in} \big(H^{1/2}(\Gamma), \| \cdot \|_{V^{-1}}\big),$$

$$(\tfrac{1}{2}I - K')^{-1} = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}I + K')^{\ell} = 2 \sum_{\ell=0}^{\infty} (2K')^{\ell} \quad \text{in} \big(H^{-1/2}(\Gamma), \| \cdot \|_{V}\big),$$

$$(\tfrac{1}{2}I + K)^{-1} = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}I - K)^{\ell} = 2 \sum_{\ell=0}^{\infty} (2K)^{\ell} \quad \text{in} \big(H^{1/2}(\Gamma)/\gamma\mathfrak{R}, \| \cdot \|_{V^{-1}}\big)$$

$$(\tfrac{1}{2}I + K')^{-1} = \sum_{\ell=0}^{\infty} (\tfrac{1}{2}I - K)^{\ell} = 2 \sum_{\ell=0}^{\infty} (-2K')^{\ell} \, \text{in} \big(H_0^{-1/2}(\Gamma), \| \cdot \|_{V}\big).$$

The operators on the left are positive definite and those on the right are contractions in the corresponding spaces.

Steinbach + Wd 2001 , M. Costabel 2006

$$
\begin{aligned}
\left\langle V^{-1}\left(\tfrac{1}{2}I \underset{(-)}{+} K\right)\mu, \mu\right\rangle \quad &\leq q^2 \langle V^{-1}\mu, \mu\rangle\ ,\ \mu \in H^{\frac{1}{2}}_{(0)}(\Gamma)\,, \\
\left\langle V\left(\tfrac{1}{2}I \underset{(-)}{+} K^*\right)\varrho, \varrho\right\rangle \quad &\leq q^2 \langle V\varrho, \varrho\rangle \quad\ ,\ \varrho \in H^{-\frac{1}{2}}_{(0)}(\Gamma)\,, \\
q = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - c_1^V c_1^D} \quad &< 1\,.
\end{aligned}
$$

(C.F. Gauss 1837, H. Poincaré 1896)

Galerkin methods, Example: Dirichlet problem for the Laplacian:

$$\mu(y) = \sum_{\ell=1}^{N} c_\ell \Psi_\ell(y) \in \mathfrak{X}_h \subset \mathfrak{X},$$

$$u_h(x) = -\frac{2^{1-n}}{\pi} \sum_{\ell=1}^{N} \int_{\Gamma\setminus\{x\}} \frac{c_\ell \Psi_\ell(y) n(y)\cdot(y-x)}{|x-y|^n} do_\Gamma(y),$$

$$\mu_{k+1} = \Big(\frac{1}{2} I + K\Big)\mu_k + \varphi.$$

$$\sum_{\ell=1}^{N} c_{\ell,k+1} \int_\Gamma \Psi_\ell(x)\Psi_t(x) do_\Gamma(x)$$

$$= -\frac{2^{1-n}}{\pi} \sum_{\ell=1}^{N} c_{\ell,k} \int_\Gamma \int_{y\in\Gamma\setminus\{x\}} \frac{\Big(\Psi_\ell(y)-\Psi_\ell(x)\Big) n(y)\cdot(y-x)}{|x-y|^n} \Psi_t(x) do_\Gamma(y) do_\Gamma(x)$$

$$+ \int_\Gamma \varphi(x)\Psi_t(x) do_\Gamma(x)\ ,\ t = 1, \dots, N; k \in \mathbb{N}.$$

Then $\lim_{k\to\infty} c_{\ell,k} = c_\ell$ and

$$\|u - u_h\|_{H^{1/2}(\Gamma)} \leq C \inf_{\widetilde{u}_h \in \mathfrak{X}_h} \|u - \widetilde{u}_h\|_{H^{1/2}(\Gamma)}$$ (Cea's lemma).

Conclusions:

- For equations of the first kind with V, respectively D, boundary Galerkin methods are stable and convergent in $H^{-\frac{1}{2}}(\Gamma)$, respectively in $H^{\frac{1}{2}}(\Gamma)$.
- Reduction of Galerkin weights to fully discrete approximation: See e.g. A.V. Setukha+A.V. Semenova 2017.
- The **Neumann series** converge in $H^{\frac{1}{2}}_{(0)}(\Gamma)$ and $H^{-\frac{1}{2}}_{(0)}(\Gamma)$, respectively.
- Discrete version with finite differences: J.T. Beale+W. Ying 2013.
- The Neumann series can be used for a posteriori error estimation.
- Boundary element Petrov Galerkin methods (corresponding to preconditioning) for equations of the first kind are also stable and convergent.
- Fast versions, in particular multipole and also wavelet methods are now available for all these boundary integral equations corresponding to Δ , $\Delta + k^2$ the Lamé system, the Stokes problem, D. Medkova 2011, and the Maxwell system, Costabel 1991.
- Efficient combination with domain decompositions in BETI: U. Langer + O. Steinbach 2003, G. Of et al. 2004 – now.

Thank you for your attention!

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