Seminar 9

1. Evaluate the following double integrals:

a) $\displaystyle\int_1^6 \int_2^3 \frac{1}{(x+y)^2}\,\mathrm{d}x\mathrm{d}y;$

b) $\displaystyle\int_0^1 \int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}}\,\mathrm{d}x\mathrm{d}y;$

c) $\displaystyle\int_0^\pi \int_0^{\pi/2} \frac{x\sin x}{(1+\cos^2 x)(1+\cos^2 y)}\,\mathrm{d}x\mathrm{d}y;$

d) **(Homework)** $\displaystyle\int_0^{\pi/2} \int_0^{\pi/2} \frac{y\sin y}{(1+\cos x)(1+\cos y)^2}\,\mathrm{d}x\mathrm{d}y;$

e) $\displaystyle\int_0^1 \int_0^1 \min\{x,y\}\,\mathrm{d}x\mathrm{d}y;$

f) $\displaystyle\int_{1/a}^a \int_0^1 \frac{1}{x^2+y^2}\,\mathrm{d}x\mathrm{d}y$, where $a > 1$.

2. Evaluate the following triple integrals:

a) $\displaystyle\int_1^2 \int_1^2 \int_1^2 \frac{1}{(x+y+z)^3}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z;$

b) **(Temă)** $\displaystyle\int_0^a \int_0^b \int_0^c (x+y+z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where $a,b,c > 0$;

c) $\displaystyle\int_0^1 \int_0^1 \int_0^1 \min\{x,y,z\}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

3. Evaluate $\displaystyle\iint_A \frac{1}{y+1}\,\mathrm{d}x\mathrm{d}y$, if A is the plane set bounded by the parabola $y = x^2$ and by the line $y = 2x+3$.

4. Evaluate $\displaystyle\iint_A \frac{x}{y^2+1}\,\mathrm{d}x\mathrm{d}y$, if A is the plane set bounded by the lines $x = \sqrt{3}$, $y = x$ and by the hyperbola $xy = 1$.

5. Compute $\displaystyle\iint_A \frac{x}{1+y^2}\,\mathrm{d}x\mathrm{d}y$, if A is the set defined by $A := \{\,(x,y) \in \mathbb{R}^2 \mid x \geq y \geq 0, x^2+y^2 \leq 2\}$.

6. **(Homework)** Compute $\displaystyle\iint_A \frac{y}{1+x^2}\,\mathrm{d}x\mathrm{d}y$, if A is the set defined by $A := \{\,(x,y)\in\mathbb{R}^2 \;\mid\; y \geq |x|, x^2+y^2 \leq 2\}$.

7. Let $a, b > 0$ and let

$$A := \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} - \frac{y^2}{b^2} \leq 1,\ -b \leq y \leq b \right\}.$$

Evaluate $\displaystyle\iint_A \frac{x^2}{y^2+b^2}\,\mathrm{d}x\mathrm{d}y$.

8. **(Homework)** Given real numbers $a, b > 0$, evaluate

$$\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}}\mathrm{d}x\mathrm{d}y.$$

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9. Compute $\displaystyle\iiint_A \frac{1}{(x+y+z+1)^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if A is the space set bounded by the planes $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

10. Compute $\displaystyle\iiint_A (x^2+y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$, if

$$A := \{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 \leq z \leq 4\,\}.$$

Solutions

$\boxed{\textbf{1.}}$ a) By Fubini's theorem we have

$$\begin{aligned}\int_1^6\int_2^3 \frac{1}{(x+y)^2}\,\mathrm{d}x\mathrm{d}y &= \int_{x=1}^{x=6}\left(\int_{y=2}^{y=3}\frac{1}{(x+y)^2}\,\mathrm{d}y\right)\mathrm{d}x\\ &= \int_{x=1}^{x=6}\left(-\frac{1}{x+y}\Big|_{y=2}^{y=3}\right)\mathrm{d}x = \int_1^6\left(-\frac{1}{x+3}+\frac{1}{x+2}\right)\mathrm{d}x\\ &= \ln\frac{x+2}{x+3}\Big|_1^6 = \ln\frac{32}{27}\,.\end{aligned}$$

b) By Fubini's theorem we have (Pay attention to the choice of order of integration! One of the two iterated integrals is easy, the other is difficult.)

$$\begin{aligned}\int_0^1\int_0^1 \frac{x}{(1+x^2+y^2)^{3/2}}\,\mathrm{d}x\mathrm{d}y &= \int_{y=0}^{y=1}\left(\int_{x=0}^{x=1}\frac{x}{(1+x^2+y^2)^{3/2}}\,\mathrm{d}x\right)\mathrm{d}y\\ &= \int_{y=0}^{y=1}\left(\frac{1}{2}\int_{x=0}^{x=1}(1+x^2+y^2)^{-3/2}\cdot 2x\,\mathrm{d}x\right)\mathrm{d}y\\ &= \int_{y=0}^{y=1}\left(\frac{1}{2}\cdot\frac{(1+x^2+y^2)^{-1/2}}{-1/2}\Bigg|_{x=0}^{x=1}\right)\mathrm{d}y\\ &= \int_{y=0}^{y=1}\left(-\frac{1}{\sqrt{1+x^2+y^2}}\Bigg|_{x=0}^{x=1}\right)\mathrm{d}y\\ &= \int_0^1\left(-\frac{1}{\sqrt{y^2+2}}+\frac{1}{\sqrt{y^2+1}}\right)\mathrm{d}y\\ &= \ln\frac{y+\sqrt{y^2+1}}{y+\sqrt{y^2+2}}\Bigg|_0^1 = \ln\frac{2+\sqrt{2}}{1+\sqrt{3}}\,.\end{aligned}$$

c) Let $I := \displaystyle\int_0^\pi\int_0^{\pi/2}\frac{x\sin x}{(1+\cos^2 x)(1+\cos^2 y)}\,\mathrm{d}x\mathrm{d}y$. By Fubini's theorem we have

$$
\begin{aligned}
I &= \int_{x=0}^{x=\pi} \left(\int_{y=0}^{y=\pi/2} \frac{x \sin x}{(1+\cos^2 x)(1+\cos^2 y)}\, \mathrm{d}y \right) \mathrm{d}x \\
&= \int_{x=0}^{x=\pi} \left(\frac{x \sin x}{1+\cos^2 x} \underbrace{\int_{y=0}^{y=\pi/2} \frac{\mathrm{d}y}{1+\cos^2 y}}_{I_2} \right) \mathrm{d}x \\
&= I_2 \underbrace{\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, \mathrm{d}x}_{I_1} = I_1 I_2.
\end{aligned}
$$

To evaluate the integral $I_1 = \displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, \mathrm{d}x$, we use the change of variable $x = \pi - t$. We obtain

$$
\begin{aligned}
I_1 &= \int_{\pi}^{0} \frac{(\pi - t)\sin(\pi - t)}{1+\cos^2(\pi - t)}\, (-\mathrm{d}t) = \int_0^{\pi} \frac{(\pi - t)\sin t}{1+\cos^2 t}\, \mathrm{d}t \\
&= \pi \int_0^{\pi} \frac{\sin t}{1+\cos^2 t}\, \mathrm{d}t - \int_0^{\pi} \frac{t \sin t}{1+\cos^2 t}\, \mathrm{d}t,
\end{aligned}
$$

whence

$$
2I_1 = \pi \int_0^{\pi} \frac{\sin t}{1+\cos^2 t}\, \mathrm{d}t = -\pi \operatorname{arctg}(\cos t)\Big|_0^{\pi} = \frac{\pi^2}{2}\,,
$$

hence $I_1 = \pi^2/4$.

To evaluate the integral $I_2 = \displaystyle\int_0^{\pi/2} \frac{\mathrm{d}y}{1+\cos^2 y}$, we use the change of variable $\operatorname{tg} y = t$. We obtain

$$
\begin{aligned}
I_2 &= \int_0^{\frac{\pi}{2}-0} \frac{\mathrm{d}y}{1+\cos^2 y} = \int_0^{\infty} \frac{1}{1+\frac{1}{1+t^2}} \cdot \frac{1}{1+t^2}\, \mathrm{d}t \\
&= \int_0^{\infty} \frac{\mathrm{d}t}{t^2+2} = \frac{1}{\sqrt{2}}\ \operatorname{arctg} \frac{t}{\sqrt{2}}\ \Big|_0^{\infty} = \frac{\pi}{2\sqrt{2}}\,.
\end{aligned}
$$

Finally, we get $I = I_1 I_2 = \dfrac{\pi^3}{8\sqrt{2}}\,.$

Remark. In the previous solution we used the following result: given two continuous functions $f : [a, b] \to \mathbb{R}$ and $g : [c, d] \to \mathbb{R}$, one has

$$\int_a^b \int_c^d f(x)g(y)\,\mathrm{d}x\mathrm{d}y = \left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_c^d g(y)\,\mathrm{d}y\right).$$

This result will often be used in other problems.

d) Answer: $\dfrac{\pi}{2} - 1$.

e) By Fubini's theorem we have

$$I := \int_0^1 \int_0^1 \min\{x, y\}\mathrm{d}x\mathrm{d}y = \int_{y=0}^{y=1} \underbrace{\left(\int_{x=0}^{x=1} \min\{x, y\}\mathrm{d}x\right)}_{G(y)}\mathrm{d}y = \int_0^1 G(y)\,\mathrm{d}y.$$

Note that

$$\begin{aligned} G(y) &= \int_{x=0}^{x=y} \min\{x, y\}\mathrm{d}x + \int_{x=y}^{x=1} \min\{x, y\}\mathrm{d}x = \int_0^y x\mathrm{d}x + \int_y^1 y\mathrm{d}x \\ &= \frac{x^2}{2}\Big|_{x=0}^{x=y} + xy\Big|_{x=y}^{x=1} = \frac{y^2}{2} + y - y^2 = y - \frac{y^2}{2}, \end{aligned}$$

hence $I = \displaystyle\int_0^1 \left(y - \frac{y^2}{2}\right)\mathrm{d}y = \frac{1}{3}$.

f) By Fubini's theorem we have

$$\begin{aligned} I &:= \int_{1/a}^a \int_0^1 \frac{1}{x^2 + y^2}\,\mathrm{d}x\mathrm{d}y = \int_{x=1/a}^{x=a} \left(\int_{y=0}^{y=1} \frac{1}{x^2 + y^2}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{x=1/a}^{x=a} \frac{1}{x}\,\mathrm{arctg}\,\frac{y}{x}\Big|_{y=0}^{y=1}\mathrm{d}x = \int_{1/a}^a \frac{1}{x}\,\mathrm{arctg}\,\frac{1}{x}\,\mathrm{d}x. \end{aligned}$$

Substituting $x = 1/t$, we obtain

$$\begin{aligned} I &= \int_a^{1/a} t\,\mathrm{arctg}\,t\left(-\frac{1}{t^2}\right)\mathrm{d}t = \int_{1/a}^a \frac{1}{t}\,\mathrm{arctg}\,t\,\mathrm{d}t \\ &= \int_{1/a}^a \frac{1}{t}\left(\frac{\pi}{2} - \mathrm{arctg}\,\frac{1}{t}\right)\mathrm{d}t = \frac{\pi}{2}\int_{1/a}^a \frac{1}{t}\,\mathrm{d}t - I, \end{aligned}$$

whence $I = \dfrac{\pi}{4} \displaystyle\int_{1/a}^{a} \frac{1}{t}\,\mathrm{d}t = \frac{\pi}{2}\,\ln a$.

$\boxed{\textbf{2.}}$ a) By Fubini's theorem we have

$$
\begin{aligned}
\int_1^2\int_1^2\int_1^2 \frac{1}{(x+y+z)^3}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_1^2\left(\int_1^2\left(\int_1^2 (x+y+z)^{-3}\,\mathrm{d}z\right)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_1^2\left(\int_1^2\left(\frac{(x+y+z)^{-2}}{-2}\Bigg|_{z=1}^{z=2}\right)\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{2}\int_1^2\left(\int_1^2\left(-\frac{1}{(x+y+2)^2}+\frac{1}{(x+y+1)^2}\right)\mathrm{d}y\right)\mathrm{d}x \\
&= \frac{1}{2}\int_1^2\left(\frac{1}{x+y+2}-\frac{1}{x+y+1}\right)\Bigg|_{y=1}^{y=2}\mathrm{d}x \\
&= \frac{1}{2}\int_1^2\left(\frac{1}{x+4}-\frac{2}{x+3}+\frac{1}{x+2}\right)\mathrm{d}x = \frac{1}{2}\,\ln\frac{(x+2)(x+4)}{(x+3)^2}\Bigg|_1^2 \\
&= \frac{1}{2}\,\ln\frac{128}{125}\,.
\end{aligned}
$$

b) Answer: $\dfrac{abc(a+b+c)}{2}$.

c) Let $I := \displaystyle\int_0^1\int_0^1\int_0^1 \min\{x,y,z\}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$. By Fubini's theorem we have

$$
I = \int_0^1\int_0^1 F_1(x,y)\,\mathrm{d}x\mathrm{d}y,
$$

where $F_1 : [0,1]\times[0,1] \to \mathbb{R}$ is the function defined by

$$
\begin{aligned}
F_1(x,y) &= \int_0^1 \min\{x,y,z\}\,\mathrm{d}z \\
&= \int_0^{\min\{x,y\}} \min\{x,y,z\}\,\mathrm{d}z + \int_{\min\{x,y\}}^1 \min\{x,y,z\}\,\mathrm{d}z \\
&= \int_0^{\min\{x,y\}} z\,\mathrm{d}z + \int_{\min\{x,y\}}^1 \min\{x,y\}\,\mathrm{d}z \\
&= \min\{x,y\} - \frac{\min\{x,y\}^2}{2}\,.
\end{aligned}
$$

Applying once again Fubini's theorem, we obtain

$$I = \int_0^1 \left(\int_0^1 \left(\min\{x,y\} - \frac{\min\{x,y\}^2}{2} \right) \mathrm{d}y \right) \mathrm{d}x.$$

But

$$\begin{aligned} &\int_0^1 \left(\min\{x,y\} - \frac{\min\{x,y\}^2}{2} \right) \mathrm{d}y \\ &\quad = \int_0^x \left(y - \frac{y^2}{2} \right) \mathrm{d}y + \int_x^1 \left(x - \frac{x^2}{2} \right) \mathrm{d}y = x - x^2 + \frac{x^3}{3}, \end{aligned}$$

whence

$$I = \int_0^1 \left(x - x^2 + \frac{x^3}{3} \right) \mathrm{d}x = \frac{1}{4}.$$

$\boxed{\textbf{3.}}$ The parabola $y = x^2$ and the line $y = 2x + 3$ intersect at two points, namely $M(-1,1)$ and $N(3,9)$ (see figure 1).

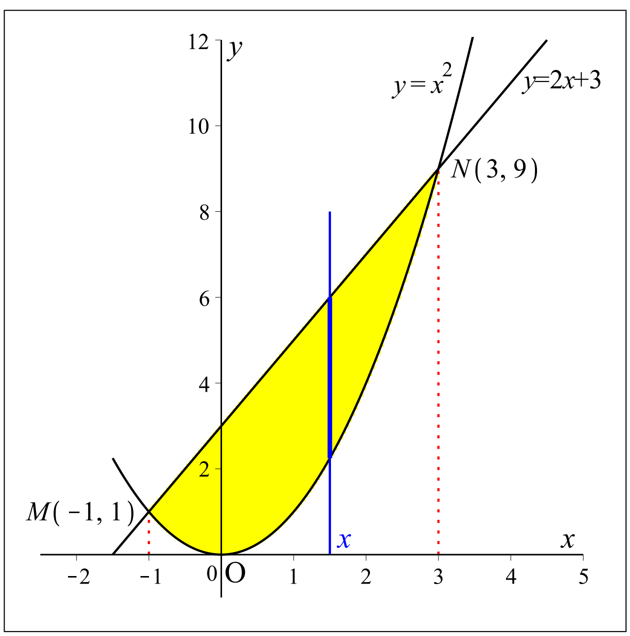

Figura 1:

By Fubini's theorem we have

$$\begin{aligned}
\iint_A \frac{1}{y+1}\,\mathrm{d}x\mathrm{d}y &= \int_{x=-1}^{x=3}\left(\int_{y=x^2}^{y=2x+3}\frac{1}{y+1}\,\mathrm{d}y\right)\mathrm{d}x\\
&= \int_{x=-1}^{x=3} \ln(y+1)\Big|_{y=x^2}^{y=2x+3}\mathrm{d}x\\
&= \int_{-1}^{3}\big(\ln(2x+4)-\ln(x^2+1)\big)\mathrm{d}x\\
&= 4+2\ln 5-\frac{\pi}{2}-2\,\mathrm{arctg}\,3.
\end{aligned}$$

$\boxed{\mathbf{4.}}$ The hyperbola $xy=1$ and the line $y=x$ intersect in the first quadrant at the point $(1,1)$ (see figure 2).

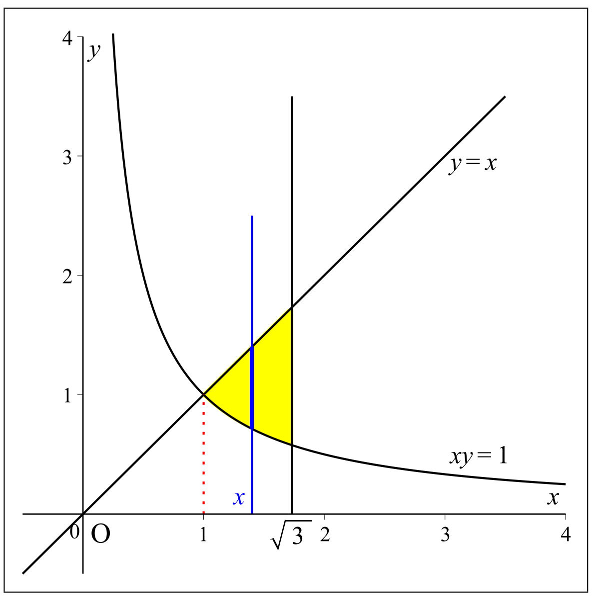

Figura 2:

Passing to iterated integrals, we have

$$\begin{aligned}
\iint_A \frac{x}{y^2+1}\,\mathrm{d}x\mathrm{d}y &= \int_{x=1}^{x=\sqrt{3}}\left(\int_{y=\frac{1}{x}}^{y=x}\frac{x}{y^2+1}\,\mathrm{d}y\right)\mathrm{d}x\\
&= \int_{1}^{\sqrt{3}} x\arctan y\Big|_{y=1/x}^{y=x}\mathrm{d}x = \int_1^{\sqrt{3}} x\left(\arctan x-\arctan\frac{1}{x}\right)\mathrm{d}x.
\end{aligned}$$

Tacking into account that $\arctan \dfrac{1}{x} = \dfrac{\pi}{2} - \arctan x$, we get

$$\iint_A \frac{x}{y^2+1}\,\mathrm{d}x\mathrm{d}y = \int_1^{\sqrt{3}} x\left(2\arctan x - \frac{\pi}{2}\right)\mathrm{d}x = 1 - \sqrt{3} + \frac{\pi}{3}\,.$$

$\boxed{\textbf{5.}}$ Passing to iterated integrals, we have (see figure 3):

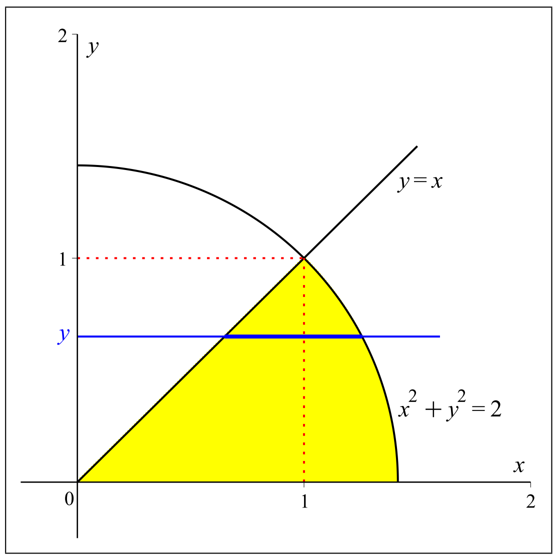

Figura 3:

$$\begin{aligned}
\iint_A \frac{x}{y^2+1}\,\mathrm{d}x\mathrm{d}y &= \int_{y=0}^{y=1}\left(\int_{x=y}^{x=\sqrt{2-y^2}} \frac{x}{y^2+1}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y=0}^{y=1} \frac{1}{y^2+1}\cdot \left.\frac{x^2}{2}\right|_{x=y}^{x=\sqrt{2-y^2}}\,\mathrm{d}y \\
&= \int_0^1 \frac{1-y^2}{y^2+1}\,\mathrm{d}y = \frac{\pi}{2} - 1.
\end{aligned}$$

$\boxed{\textbf{6.}}$ Answer: $\pi - 2$.

$\boxed{\textbf{7.}}$ Passing to iterated integrals, we have (see figure 4):

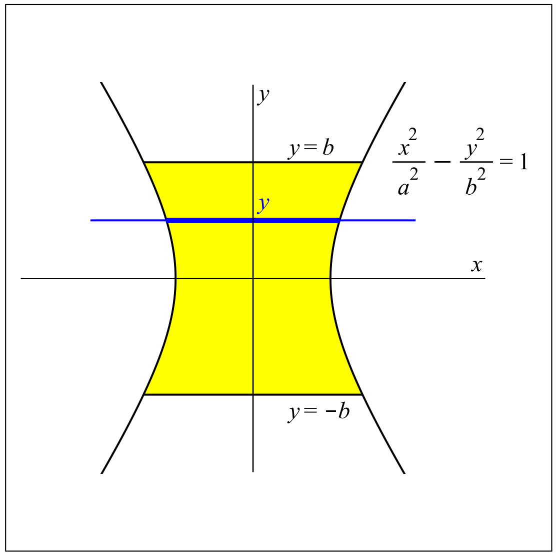

Figura 4:

$$\begin{aligned}
\iint_A \frac{x^2}{y^2+b^2}\,\mathrm{d}x\mathrm{d}y &= \int_{y=-b}^{y=b}\left(\int_{x=-\frac{a}{b}\sqrt{y^2+b^2}}^{x=\frac{a}{b}\sqrt{y^2+b^2}} \frac{x^2}{y^2+b^2}\,\mathrm{d}x\right)\mathrm{d}y \\
&= \int_{y=-b}^{y=b} \frac{1}{y^2+b^2}\cdot\left.\frac{x^3}{3}\right|_{x=-\frac{a}{b}\sqrt{y^2+b^2}}^{x=\frac{a}{b}\sqrt{y^2+b^2}}\mathrm{d}y \\
&= \int_{-b}^{b} \frac{1}{y^2+b^2}\cdot\frac{2a^3(y^2+b^2)^{3/2}}{3b^3}\,\mathrm{d}y \\
&= \frac{2a^3}{3b^3}\int_{-b}^{b}\sqrt{y^2+b^2}\,\mathrm{d}y = \frac{2a^3}{3b}\left(\sqrt{2}+\ln(1+\sqrt{2})\right).
\end{aligned}$$

$\boxed{\textbf{8.}}$ Let I denote the integral to be evaluated. Note that $[0,a]\times[0,b] = A_1 \cup A_2$, where

$$\begin{aligned}
A_1 &= \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; 0\le x\le a,\ 0\le y\le \frac{b}{a}\,x \right\}, \\
A_2 &= \left\{ (x,y)\in\mathbb{R}^2 \;\middle|\; 0\le y\le b,\ 0\le x\le \frac{a}{b}\,y \right\}.
\end{aligned}$$

Tacking this into account, we deduce that

$$\begin{aligned} I &= \int_0^a \left(\int_0^{\frac{b}{a}x} e^{b^2x^2} \mathrm{d}y \right) \mathrm{d}x + \int_0^b \left(\int_0^{\frac{a}{b}y} e^{a^2y^2} \mathrm{d}x \right) \mathrm{d}y \\ &= \int_0^a \frac{b}{a} x e^{b^2x^2} \mathrm{d}x + \int_0^b \frac{a}{b} y e^{a^2y^2} \mathrm{d}y = \frac{e^{a^2b^2}-1}{ab}. \end{aligned}$$

$\boxed{\mathbf{9.}}$ A is the set of all points lying inside and on the faces of the tetrahedron $OMNP$, where O is the origin, while $M(1,0,0)$, $N(0,1,0)$, $P(0,0,1)$ (see figure 5). The projection of A onto the plane Oxy is the triangle OMN, i.e., the set $A_0 := \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0,\ x+y \leq 1\}$.

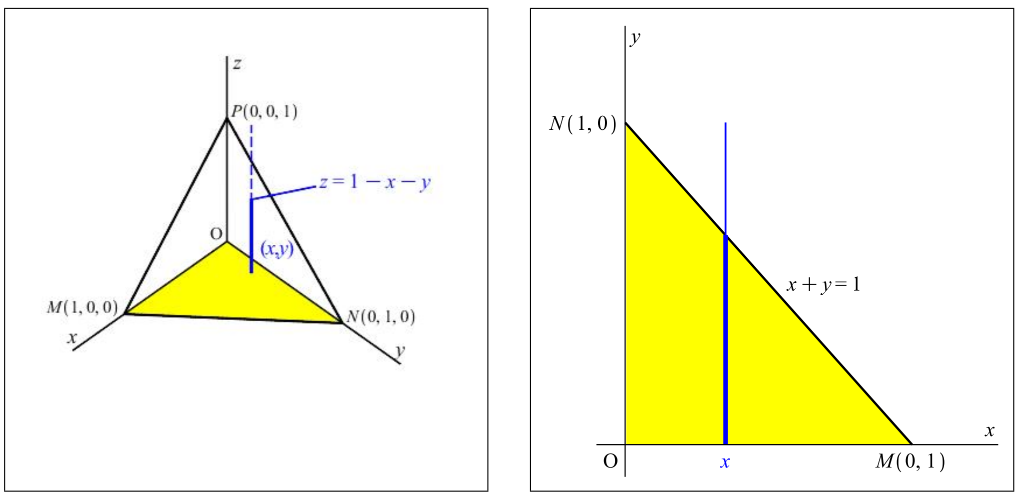

Figura 5:

Passing to iterated integrals, we have

$$\begin{aligned} &\iiint_A \frac{1}{(x+y+z+1)^2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{A_0} \left(\int_{z=0}^{z=1-x-y} \frac{\mathrm{d}z}{(x+y+z+1)^2} \right) \mathrm{d}x\mathrm{d}y \\ &\quad = \iint_{A_0} -\frac{1}{x+y+z+1}\Big|_{z=0}^{z=1-x-y} \mathrm{d}x\mathrm{d}y = \iint_{A_0} \left(-\frac{1}{2} + \frac{1}{x+y+1} \right) \mathrm{d}x\mathrm{d}y \\ &\quad = -\frac{1}{4} + \iint_{A_0} \frac{1}{x+y+1} \mathrm{d}x\mathrm{d}y, \end{aligned}$$

because $\iint_{A_0} \mathrm{d}x\mathrm{d}y = m(A_0) = \dfrac{1}{2}$. To evaluate the double integral, we use iterated integrals. We have

$$\begin{aligned}
\iint_{A_0} \frac{1}{x+y+1}\,\mathrm{d}x\mathrm{d}y &= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} \frac{1}{x+y+1}\,\mathrm{d}y \right) \mathrm{d}x \\
&= \int_{x=0}^{x=1} \ln(x+y+1)\Big|_{y=0}^{y=1-x} \mathrm{d}x = \int_0^1 \big(\ln 2 - \ln(x+1)\big)\mathrm{d}x = 1 - \ln 2.
\end{aligned}$$

Finally, we get $\displaystyle\iiint_A \frac{1}{(x+y+z+1)^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{3}{4} - \ln 2.$

$\boxed{\textbf{10.}}$ A is the set of all points lying inside the sheet of the paraboloid $z = x^2 + y^2$, below the plane $z = 4$ (see figure 6).

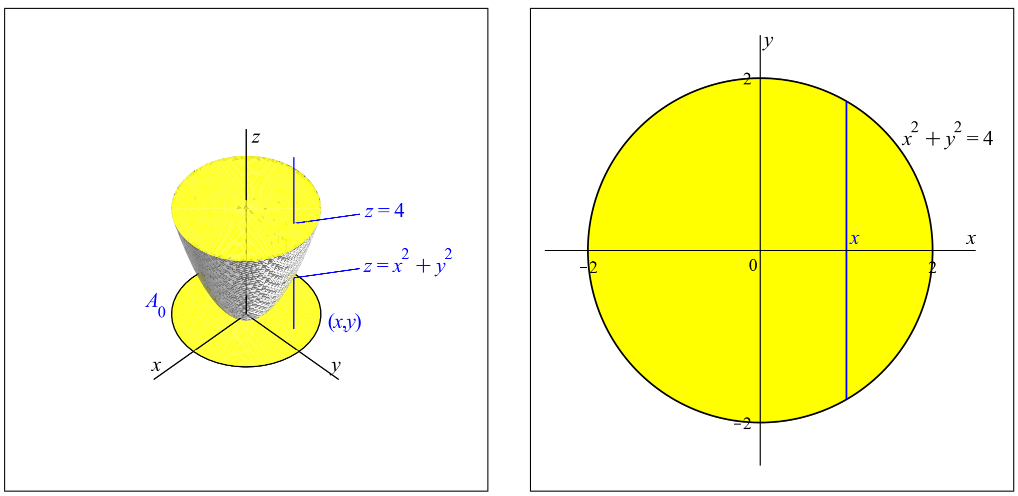

Figura 6:

The projection of A onto the plane Oxy is the disk A_0 about the origin, with radius 2. Passing to iterated integrals, we have

$$\begin{aligned}
I \quad &:= \quad \iiint_A (x^2+y^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \iint_{A_0} \left(\int_{z=x^2+y^2}^{z=4} (x^2+y^2)\,\mathrm{d}z \right) \mathrm{d}x\mathrm{d}y \\
&= \quad \iint_{A_0} (x^2+y^2)(4-x^2-y^2)\,\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Trying to evaluate the double integral by using iterated integrals, we obtain

$$I = \int_{x=-2}^{x=2} \left(\int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} (x^2+y^2)(4-x^2-y^2)\,\mathrm{d}y \right) \mathrm{d}x.$$

HORRIBLY ! (convince yourself by doing the calculations if you do not believe)

Therefore, we will calculate the double integral not with the help of Fubini's theorem, but with the help of polar coordinates (passing to polar coordinates is actually a change of variables). We have (see figure 7)

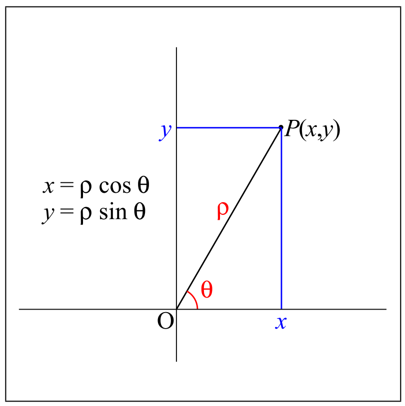

Figura 7:

$$x = \rho\cos\theta, \qquad \rho \in [0,2],$$
$$y = \rho\sin\theta, \qquad \theta \in [0,2\pi].$$

The Jacobian determinant of the coordinate conversion is

$$\frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho.$$

After changing to polar coordinates, we obtain

$$\begin{aligned} I &= \int_0^2 \int_0^{2\pi} \left(\rho^2\cos^2\theta + \rho^2\sin^2\theta\right)\left(4-\rho^2\cos^2\theta-\rho^2\sin^2\theta\right)\cdot\rho\,\mathrm{d}\rho\mathrm{d}\theta \\ &= \int_0^2 \int_0^{2\pi} \rho^3(4-\rho^2)\,\mathrm{d}\rho\mathrm{d}\theta \\ &= \left(\int_0^2 (4\rho^3-\rho^5)\,\mathrm{d}\rho\right)\left(\int_0^{2\pi} \mathrm{d}\theta\right). \end{aligned}$$

The remark after the solution to problem 1c) was used for the last equality.
Answer: $I = \dfrac{32\pi}{3}$.